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Effective five-wave Hamiltonian for surface water waves

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Abstract

An effective five-wave Hamiltonian for one-dimensional water waves on the surface of an infinitely deep ideal fluid is presented. The diagrammatic technique is used. For some particular orientations of wave vectors the interaction matrix element is exactly zero, while for others it is given by remarkably simple formulas. This is another surprising result of water wave theory. © 1997 Published by Elsevier Science B.V.

1. Introduction

In this article five-wave interactions of gravity waves on the surface of an ideal fluid of infinite depth are studied. A whole set of physical processes is described by the five-wave interactions, for instance, the $2 \Leftrightarrow 3$ instability, and the creation of horse-shoe-like structures [1,2]. An important property of such processes is that they do not conserve the wave action integral.

Five-wave interactions are of great significance in the one-dimensional case because the amplitudes of four-wave interactions in the effective Hamiltonian is exactly equal to zero [3-5] and the fifth-order interaction is the first nonvanishing term [6,4]. In the weakly two-dimensional case then, the narrow spectra are defined by the combination of one-dimensional five-wave interactions, and four-wave interactions with small angle.

The mathematical reason for the vanishing of the four-wave interaction term in the one-dimensional case is not yet understood. There was even a suggestion that such a vanishing also occurred in higher orders. Bryant and Stiassnie [7] showed that a certain fifth-order amplitude was zero. That observation may have led some to believe that all fifth-order terms were zero. This was shown to be wrong by Diachenko, Lvov and Zakharov [6] for collinear wave vector interaction, thus proving nonintegrability of the system. However, the final expression for the collinear wave vector of the fifth-order interaction is simple and compact, which is yet another unsolved problem.

In the current work, fifth-order matrix elements for all possible relative orientations of the wave vectors are obtained. For two particular orientations the resulting interactions vanish, and in all other orientations they are nonzero, but given by remarkably simple expressions.

This work is a natural continuation of Ref. [6]. The same formulation of the Hamiltonian formalism and the same symbolic definitions are used.

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2. Canonical variables and the Hamiltonian of the problem

The basic set of equations describing a two-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is the following one,

$$\phi_{xx} + \phi_{zz} = 0 \qquad (\phi_z \to 0, z \to -\infty), \qquad \eta_t + \eta_x\phi_x = \phi_z|_{z=\eta}, \qquad \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0|_{z=\eta}.$$

Here $\eta(x,t)$ is the shape of the surface, $\phi(x,z,t)$ is the potential function of the flow and g is the gravitational constant. As was shown by Zakharov in Ref. [8], the potential on the surface $\psi(x,t) = \phi(x,z,t)|_{z=\eta}$ and $\eta(x,t)$ are canonically conjugated, and their Fourier transforms satisfy the equations

$$\frac{\partial\psi_k}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta_k^*}, \qquad \frac{\partial\eta_k}{\partial t} = \frac{\delta\mathcal{H}}{\delta\psi_k^*}.$$

Here $\mathcal{H} = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms,

$$K = \tfrac{1}{2}\int \mathrm{d}x \int_{-\infty}^{\eta} v^2\,\mathrm{d}z, \qquad U = \frac{g}{2}\int \eta^2\,\mathrm{d}x.$$

A Hamiltonian can be expanded in an infinite series in powers of the characteristic wave steepness $k\eta_k \ll 1$ [8,9] by using an iterative procedure. All terms up to the fifth order of this series contribute to the amplitude of the five-wave interaction. So the Hamiltonian is expressed in terms of complex wave amplitudes a_k, a_k^* which satisfy the canonical equation of motion,

$$\frac{\partial a_k}{\partial t} + \mathrm{i}\frac{\delta\mathcal{H}}{\delta a_k^*} = 0. \tag{1}$$

Here $\omega_k = \sqrt{g|k|}$ is the dispersion law for the gravity waves. $\mathcal{H}$ can be expanded as follows,

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \dots\ . \tag{2}$$

In the normal variable a_k the second order term in the Hamiltonian acquires the form

$$\mathcal{H}_2 = \int \omega_k a_k a_k^*\,\mathrm{d}k.$$

The third order term, which describes $0 \Leftrightarrow 3$ (first term) and $1 \Leftrightarrow 2$ processes (second term) is

$$\mathcal{H}_3 = \tfrac{1}{2}\int V^{k_1}_{k_2k_3}(a^*_{k_1}a_{k_2}a_{k_3} + a_{k_1}a^*_{k_2}a^*_{k_3})\delta_{k_1-k_2-k_3}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$$
$$+ \tfrac{1}{6}\int U_{k_1k_2k_3}(a_{k_1}a_{k_2}a_{k_3} + a^*_{k_1}a^*_{k_2}a^*_{k_3})\delta_{k_1+k_2+k_3}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3.$$

The fourth order term in the Hamiltonian consists of three terms,

$$\mathcal{H}_4 = \tfrac{1}{24}\int R_{k_1k_2k_3p}(a_{k_1}a_{k_2}a_{k_3}a_p + a^*_{k_1}a^*_{k_2}a^*_{k_3}a^*_p)\delta_{k_1+k_2+k_3+p}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}p$$
$$+ \tfrac{1}{6}\int G^{k_1}_{k_2k_3p}(a^*_{k_1}a_{k_2}a_{k_3}a_p + a_{k_1}a^*_{k_2}a^*_{k_3}a^*_p)\delta_{k_1-k_2-k_3-p}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}p$$
$$+ \tfrac{1}{4}\int W^{k_1k_2}_{k_3p}a^*_{k_1}a^*_{k_2}a_{k_3}a_p\delta_{k_1+k_2-k_3-p}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}p,$$

describing different types of wave interactions (the first term $4 \Leftrightarrow 0$, the second term $3 \Leftrightarrow 1$, the last term $2 \Leftrightarrow 2$ interactions).

Among the different terms of the fifth order, the only resonant term corresponds to the process $2 \Leftrightarrow 3$,

$$\mathcal{H}_5 = \tfrac{1}{12} \int Q_{pq}^{k_1 k_2 k_3} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_p a_q + \text{c.c.}) \delta_{k_1+k_2+k_3-p-q} \, dk_1 \, dk_2 \, dk_3 \, dp \, dq$$

Here $V_{k_2,k_3}^{k_1}, U_{k_1,k_2,k_3}, R_{k_1 k_2 k_3 p}, G_{k_2 k_3 p}^{k_1}, W_{k_3 p}^{k_1 k_2}, Q_{pq}^{k_1 k_2 k_3}$ are interaction matrix elements of the third, fourth and fifth order.

The Hamiltonian $\mathcal{H}$ in the normal variables a_k is too complicated to work with. Our purpose is to simplify the Hamiltonian to the form

$$\mathcal{H} = \int \omega_k b_k b_k^* \, dk + \tfrac{1}{4} \int T_{k_3 p}^{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_3} b_p \delta_{k_1+k_2-k_3-p} \, dk_1 \, dk_2 \, dk_3 \, dp$$
$$+ \tfrac{1}{12} \int T_{pq}^{k_1 k_2 k_3} (b_{k_1}^* b_{k_2}^* b_{k_3}^* b_p b_q + \text{c.c.}) \delta_{k_1+k_2+k_3-p-q} \, dk_1 \, dk_2 \, dk_3 \, dp \, dq. \tag{3}$$

One of the ways to do that is to perform a canonical transformation [10,11]

$$a_k = b_k + \int \Gamma_{k_1 k_2}^{k} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - 2 \int \Gamma_{k k_1}^{k_2} b_{k_1}^* b_{k_2} \delta_{k+k_1-k_2}$$
$$+ \int \Gamma_{k k_1 k_2} b_{k_1}^* b_{k_2}^* \delta_{k+k_1+k_2} + \int B_{k_2 k_3}^{k k_1} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} + \ldots, \tag{4}$$

where the Γ's and B's are determined in such a way that the transformation is canonical, and that the transformed Hamiltonian has the form (3). The transformation (4) is canonical up to terms of order $|b_k|^3$.

On the resonant manifold $\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}$, $\boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$ there are two types of resonances - trivial and nontrivial. Trivial resonances are $k_2 = k_1$, $k_3 = k$, or $k_3 = k_1$, $k_2 = k$. Nontrivial resonances may be parameterized as

$$k = a(1+\zeta)^2, \qquad k_1 = a(1+\zeta)^2\zeta^2, \qquad k_2 = -a\zeta^2, \qquad k_3 = a(1+\zeta+\zeta^2)^2. \tag{5}$$

It was shown in Refs. [3-5] that on the nontrivial manifold (5) $T_{k_2,k_3}^{k,k_1} \equiv 0$, i.e. four-wave processes do not produce "new wave vectors", and that system is integrable to this degree of accuracy. This was the main motivation for investigating fifth-order interactions.

To find $T_{k_3 p}^{k k_1 k_2}$ one can calculate the terms of the order of b^3 and b^4 in the canonical transformation (4). This very cumbersome procedure was performed by Krasitskii [12], but the resulting expressions are so complicated that they can hardly be used for any practical purpose. Here the method of Feynman diagrams presented in [13,6] is used.

First one introduces the so-called formal classical scattering matrix which relates the asymptotic states of the system "before" and "after" interactions, $c_k^+ = \hat{S}[c_k^-]$.

$\hat{S}[c_k^-]$ is a nonlinear operator which can be presented as a series in powers of c^-, c^{-*}. It has the following form,

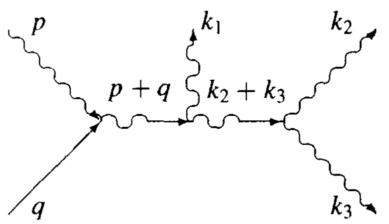
$$\hat{S}[c_k^-] = c_k^- - \sum_{n+m\geqslant 3} \frac{2\pi \mathrm{i}}{(n-1)!m!} \int S_{nm}(k, k_1, \ldots, k_{n-1}; k_n, \ldots, k_{n+m-1}) \delta_{k+k_1+\ldots+k_{n-1}-k_n-\ldots-k_{n+m-1}}$$
$$\times \delta_{\omega_k+\omega_{k_1}+\ldots+\omega_{k_{n-1}}-\omega_{k_n}-\ldots-\omega_{k_{n+m-1}}} c^{-*}_{k_1} \ldots c^{-*}_{k_{n-1}} c^-_{k_n} \ldots c^-_{k_{n+m-1}} \, dk_1 \ldots dk_{n+m-1}. \tag{6}$$

We will treat this series as a formal one and will not care about its convergence [6,14]. The functions S_{nm} are the elements of the scattering matrix. They are defined on the resonant manifolds

$$\boldsymbol{k}+\boldsymbol{k}_1+\ldots+\boldsymbol{k}_{n-1}=\boldsymbol{k}_n+\ldots+\boldsymbol{k}_{n+m-1}, \qquad \omega_k+\omega_{k_1}+\ldots+\omega_{k_{n-1}}=\omega_{k_n}+\ldots+\omega_{k_{n+m-1}}. \tag{7}$$

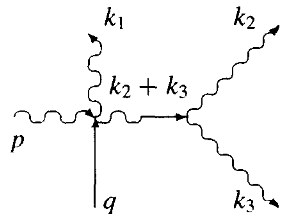
Note that the value of the matrix element S_{nm} on the resonant manifold (7) is invariant with respect to the canonical transformation (4) and that there is a simple algorithm for the calculation of the matrix elements. The element S_{nm} is a finite sum of terms which can be expressed through the coefficients of the Hamiltonians $H_i, i \leqslant n+m$. Each term can be marked by a certain Feynman diagram taken in a "tree" approximation, i.e. having no internal loops. To calculate $T_{k_3 p}^{kk_1k_2}$ one calculates the first nonzero elements of the scattering matrix for the Hamiltonian (2) and for the Hamiltonian (3). Because these two Hamiltonians are connected by the canonical transformation (4), the results must coincide. The first nontrivial element of the scattering matrix in the one-dimensional case is $S_{32}(k,k_1,k_2,k_3,p)=T_{k_3 p}^{kk_1k_2}$.

If $S_{32}(k,k_1,k_2,k_3,p)$ is calculated in terms of the initial Hamiltonian (2) it consists of 81 terms, with 60 diagrams combining three third order interactions. One of such diagrams with the corresponding expressions is



$$\frac{V_{k_2,k_3}^{k_2+k_3}\, V_{k_1,k_2+k_3}^{p+q}\, V_{p,q}^{p+q}}{\left(\omega_{k_2}+\omega_{k_3}-\omega_{k_2+k_3}\right)\left(\omega_p+\omega_q-\omega_{p+q}\right)}$$

It also has 20 diagrams combining third- and fourth-order interactions. One of such diagrams with the corresponding expressions is



$$\frac{V_{k_2,k_3}^{k_2+k_3}\, W_{k_1,k_2+k_3}^{p,q}}{\omega_{k_2}+\omega_{k_3}-\omega_{k_2+k_3}}$$

It also has the fifth-order vertex itself.

We use "Mathematica 2.2" for performing the analytical and numerical calculations of this paper. Initially, the expression for $T_{3,4,5}^{1,2}$ occupies 1 megabyte of computer memory, but we were able to simplify it to the form presented below. For some of the orientations, $T_{345}^{1,2}$ is equal to zero. We verify this fact by computing T_{345}^{12} numerically on 100 random points of the resonant manifold (8) and get zero with an accuracy of 10^{-90}.

3. Results

We present here the results of the calculation of the matrix element on the resonant manifold,

$$k_1+k_2+k_3=p+q, \qquad \omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega_p+\omega_q, \qquad \omega_k=\sqrt{g|k|}. \tag{8}$$

There are five topologically different configurations for the $k_1,k_2,k_3 \to p,q$ interaction on (8) for arbitrary signs of the wave vectors in 1D:

(i) All wave vectors positive.
(ii) Positive p and q, and one of the k_1,k_2,k_3 negative.
(iii) Positive p and q, and two of the k_1,k_2,k_3 negative.

Table 1
Results of the calculations for configurations (i)–(v)

Parametrization	Relation	Matrix element value
(i) all vectors positive		
$\omega_1 = c(a^2 - b^2 + 1 - 2a)$		$T_{pq}^{k_1 k_2 k_3} = (2/g^{1/2}\pi^{3/2})\sqrt{\omega_{k_1}\omega_{k_2}\omega_{k_3}/\omega_p\omega_q}$
$\omega_2 = c(a^2 - b^2 + 1 + 2a)$		$\times k_1 k_2 k_3 pq/\max(k_1, k_2, k_3)$
$\omega_3 = 4c$		$= (2/g^{1/2}\pi^{3/2})\omega_1^{5/2}\omega_2^{5/2}\omega_3^{5/2}\omega_p^{3/2}\omega_q^{3/2}/\max(\omega_1^2, \omega_2^2, \omega_3^2)g^4$
$\omega_p = c(a^2 - b^2 + 3 - 2b)$		
$\omega_q = c(a^2 - b^2 + 3 + 2b)$		
$c > 0,\ 0 < a, b < 1,\ \lvert a \pm b\rvert < 1$		
(ii) positive p and q, and one of k_1, k_2, k_3 negative, choose $k_1 > k_2$		
$\omega_1 = c(a + ab + b),\ k_1 \propto \omega_1^2$	$\omega_1 > \omega_p, \omega_q$	$T_{p,q}^{k_1 k_2 k_3} = (1/g^{9/2}\pi^{3/2})\omega_{k_1}^{3/2}\omega_{k_2}^{11/2}\omega_{k_3}^{1/2}\omega_p^{1/2}\omega_q^{1/2}$
$\omega_2 = c(ab - 1), k_2 \propto \omega_2^2$	$\omega_p, \omega_q > \omega_3 > \omega_2$	
$\omega_3 = c, k_3 \propto -\omega_3^2$		
$\omega_p = c(a + 1)b, p \propto \omega_p^2$	$\omega_1 > \omega_p, \omega_q$	$T_{pq}^{k_1 k_2 k_3} = (1/g^{9/2}\pi^{3/2})\omega_{k_1}^{3/2}\omega_{k_2}^{3/2}\omega_{k_3}^{5/2}\omega_p^{1/2}\omega_q^{1/2}(2\omega_{k_2}^2 - \omega_{k_3}^2)$
$\omega_q = c(b + 1)a, q \propto \omega_q^2$	$\omega_p, \omega_q > \omega_2 > \omega_3$	
$a, b > 0, ab > 1$		
(iii) positive p and q, and two of k_1, k_2, k_3 negative		zero
(iv) p and q have different signs, k_1, k_2, k_3 positive		zero
(v) p and q have different signs, and one of k_1, k_2, k_3 negative, choose $k_1 > k_2$		
$\omega_p = \omega_1 + (\omega_2^2 + \omega_2\omega_3)/(\omega_1 + \omega_2 + \omega_3)$	$\omega_1 > \omega_2 > \omega_3$	$T_{p,q}^{k_1,k_2,k_3} = -(1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{11/2}\omega_p^{3/2}\omega_q^{1/2}$
$\omega_q = (\omega_1 + \omega_3)(\omega_2 + \omega_3)/(\omega_1 + \omega_2 + \omega_3)$	$\omega_p > \omega_q$	
$k_1 = \omega_1^2/g$		
$k_2 = \omega_2^2/g$	$\omega_1 > \omega_3 > \omega_2$	$T_{p,q}^{k_1,k_2,k_3} = (1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{5/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}(\omega_{k_2}^2 - 2\omega_{k_3}^2)$
$k_3 = -\omega_3^2/g$	$\omega_p > \omega_q$	
$p = -\omega_p^2/g$		
$q = \omega_q^2/g$	$\omega_3 > \omega_1 > \omega_2$	$T_{p,q}^{k_1,k_2,k_3} = (1/\pi^{3/2}g^{9/2})\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}$
	$\omega_4 > \omega_5$	$\times(\omega_1^4 + \omega_2^4 - 2\omega_1^2\omega_3^2 - 2\omega_2^2\omega_3^2 + \omega_3^4)$
	$\omega_3 > \omega_1 > \omega_{k_2}$	$T_{p,q}^{k_1,k_2,k_3} = -(1/\pi^{3/2}g^{9/2})2\omega_1^{5/2}\omega_2^{5/2}\omega_3^{3/2}\omega_p^{3/2}\omega_q^{1/2}$
	$\omega_q > \omega_p$	

(iv) p, q with different signs, k_1, k_2, k_3 positive.
(v) p, q with different signs, and one of k_1, k_2, k_3 negative.

The results of these calculations are presented below, and summarized in Table 1. Some of the final expressions are naturally less symmetric than others, because the symmetry of expression reflects the symmetry of the wave-vector setup.

(i) The solution is given in Ref. [6],

$$T_{pq}^{k_1k_2k_3} = \frac{2}{g^{1/2}\pi^{3/2}}\sqrt{\frac{\omega_{k_1}\omega_{k_2}\omega_{k_3}}{\omega_p\omega_q}}\frac{k_1k_2k_3pq}{\max(k_1,k_2,k_3)} = \frac{2}{g^{1/2}\pi^{3/2}}\frac{\omega_1^{5/2}\omega_2^{5/2}\omega_3^{5/2}\omega_p^{3/2}\omega_q^{3/2}}{\max(\omega_1^2,\omega_2^2,\omega_3^2)g^4}. \tag{9}$$

(ii) Choose k_3 to be negative, and $k_1 > k_2$. One can parameterize the resonant manifold by

$$\begin{aligned}&\omega_1 = c(a+ab+b), \quad k_1 = \omega_1^2/g, \quad \omega_2 = c(ab-1), \quad k_2 = \omega_2^2/g,\\ &\omega_3 = c, \quad k_3 = -\omega_3^2/g, \quad \omega_p = c(a+1)b, \quad p = \omega_p^2/g,\\ &\omega_q = c(b+1)a, \quad q = \omega_q^2/g,\end{aligned} \tag{10}$$

where $c > 0$, $a, b > 0$ with $ab > 1$.

The result depends upon the sign of $\omega_2 - \omega_3$.

(a) $\omega_2 < \omega_3$. In this case $\omega_{k_1} > \omega_p, \omega_q > \omega_{k_3} > \omega_{k_2}$,

$$T_{pq}^{k_1k_2k_3} = \frac{1}{g^{9/2}\pi^{3/2}}\omega_{k_1}^{3/2}\omega_{k_2}^{11/2}\omega_{k_3}^{1/2}\omega_p^{1/2}\omega_q^{1/2}. \tag{11}$$

(b) $\omega_2 > \omega_3$. Here $\omega_{k_1} > \omega_p, \omega_q > \omega_{k_2} > \omega_{k_3}$,

$$T_{pq}^{k_1k_2k_3} = \frac{1}{g^{9/2}\pi^{3/2}}\omega_{k_1}^{3/2}\omega_{k_2}^{3/2}\omega_{k_3}^{5/2}\omega_p^{1/2}\omega_q^{1/2}(2\omega_{k_2}^2 - \omega_{k_3}^2). \tag{12}$$

(iii) Zero.

(iv) Zero.

(v) Choose $q < 0$ and $k_3 < 0$ and positive p, k_1, k_2. Then for any positive $\omega_1, \omega_2, \omega_3$ one can find ω_p, ω_q satisfying resonant condition (8) to be

$$\omega_p = \frac{\omega_1^2 + \omega_1\omega_2 + \omega_2^2 + \omega_1\omega_3 + \omega_2\omega_3}{\omega_1 + \omega_2 + \omega_3}, \qquad \omega_q = \frac{(\omega_1 + \omega_3)(\omega_2 + \omega_3)}{\omega_1 + \omega_2 + \omega_3}.$$

One can see from this parameterization that $\omega_p > \omega_1, \omega_q > \omega_3, \omega_p > \omega_2$. Choose $\omega_{k_1} > \omega_{k_2}$. Then there are three variants of relations between $\omega_{k_1}, \omega_{k_2}, \omega_{k_3}$. One of them, $\omega_{k_3} > \omega_{k_1} > \omega_{k_2}$, does not fix the relation between ω_p and ω_q, so there are four different cases of relations between ω's for which the fifth-order matrix element can be calculated.

(a) If $\omega_1 > \omega_2 > \omega_3$ then $\omega_p > \omega_q$ and

$$T_{p,q}^{k_1,k_2,k_3} = -\frac{\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{11/2}\omega_p^{3/2}\omega_q^{1/2}}{\pi^{3/2}g^{9/2}}.$$

(b) If $\omega_1 > \omega_3 > \omega_2$ then $\omega_p > \omega_q$ and

$$T_{p,q}^{k_1,k_2,k_3} = \frac{\omega_{k_1}^{1/2}\omega_{k_2}^{5/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}(\omega_{k_2}^2 - 2\omega_{k_3}^2)}{\pi^{3/2}g^{9/2}}.$$

(c) If $\omega_{k_3} > \omega_{k_1} > \omega_{k_2}$ and $\omega_p > \omega_q >$ then

$$T_{p,q}^{k_1,k_2,k_3} = \frac{\omega_{k_1}^{1/2}\omega_{k_2}^{1/2}\omega_{k_3}^{3/2}\omega_p^{3/2}\omega_q^{1/2}(\omega_1^4 + \omega_2^4 - 2\omega_1^2\omega_3^2 - 2\omega_2^2\omega_3^2 + \omega_3^4)}{\pi^{3/2}g^{9/2}}.$$

(d) If $\omega_{k_3} > \omega_{k_1} > \omega_{k_2}$ and $\omega_q > \omega_p$, then

$$T_{p,q}^{k_1,k_2,k_3} = -\frac{2}{\pi^{3/2} g^{9/2}} \omega_1^{5/2} \omega_2^{5/2} \omega_3^{3/2} \omega_p^{3/2} \omega_q^{1/2}.$$

4. Conclusion

All the matrix elements for the fifth-order interaction in one dimension are calculated in this paper for surface waves on top of an ideal fluid of infinite depth. The expressions obtained are astonishingly simple. For some particular orientation of the wave vectors the interaction matrix element is equal to *zero*. We think that this fact has a deep physical meaning and that there should be simpler ways of getting these results.

This article answers the question *how* waves interact in the case of one-dimensional waves on the surface of an infinitely deep ideal fluid. It is still to be explained *why* they interact in such a way, and *why* some of them do not interact at all.

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