Statistical description of acoustic turbulence

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We develop expressions for the nonlinear wave damping and frequency correction of a field of random, spatially homogeneous, acoustic waves. The implications for the nature of the equilibrium spectral energy distribution are discussed. [S1063-651X(97)02606-8]

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I. INTRODUCTION AND GENERAL DISCUSSION

Weak or wave turbulence, which describes the behavior of a spatially homogeneous field of weakly interacting, random dispersive waves, has led to spectacular success in our understanding of spectral energy transfer processes in plasmas, oceans, and planetary atmospheres [1]. Furthermore, the subject provides a useful paradigm for helping one think about some of the challenges of fully developed turbulence. First and foremost, the equation for the long time behavior of the spectral cumulants (which are in one to one correspondence with the spectral moments) are *closed* without making *a priori* and unjustifiable assumptions on the statistics of the processes (such as the quasi-Gaussian approximation). Second, the kinetic equation, which describes the spectral energy transfer via n-wave resonant processes, admits classes of *exact* equilibrium solutions that can be identified as pure Kolmogorov spectra, namely, equilibria for which there is a constant spectral flux of one of the conserved densities (e.g., energy, number density) of the physical process under consideration. By contrast, the thermodynamic equilibria, which have very little relevance in any turbulence theory that must account for a sink at small scales, have zero flux. Third, the theory allows one to glimpse the origin of the intermitency and the breakdown of the conditions under which one can expect relaxation to one of the finite flux Kolmogorov equilibria.

The basic ideas for writing down the kinetic equation to describe how weakly interacting waves share their energies go back to Peierls, but the modern theories have their origin in the works of Hasselman [2], Benney and Saffmann [3], Kadomtsev [4], Zakharov [1], and Benney and Newell [5,6]. A particularly important event in this history was the discovery of the pure Kolmogorov solution by Zakharov [7]. Usually, the thermodynamic equilibrium solutions can be seen from the kinetic equation by inspection. On the other hand, the solutions, corresponding to pure Kolmogorov spectra are much more subtle and only emerge after one has exploited scaling symmetries of the dispersion relation and the coupling coefficient via what is now called the Zakharov transformation [1,7].

Success to this point, namely, the natural closure, depended crucially on the fact the waves were dispersive. This means that the group velocity is neither constant in amplitude nor direction, or that, alternatively stated, the dispersion tensor

$$D_{\alpha\beta}=\left(\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\right), \quad 0<\alpha,\beta\leqslant\alpha \tag{1.1}$$

has full rank. Here d is the system dimension, Greek letters (here α and β) denote tensor indices varying from 1 to space dimension d, and

$$\omega=\omega(\mathbf{k}) \tag{1.2}$$

is the linear dispersion relation. The reason for closure is slaving. In a field of weakly interacting random dispersive waves, the first step to slaving is achieved by the linear characteristics of the wave trains. Systems, which initially are strongly correlated, are decorrelated because different waves carry statistically independent information from long distances at different speeds and directions. Cumulants of order $N>2$ tend to zero on the fast time scale ω_{I}^{-1} (ω_{I} is a typical frequency at which the energy is injected). The system approaches a state of exact joint Gaussian statistics. The energy at each wave vector remains constant, and there is no transfer. But the systems of interest to us are nonlinear and, therefore, although the cumulants undergo an initial decay, they are regenerated by the nonlinear terms. In particular, the cumulant of the order N is regenerated both by cumulants of higher order and by products of lower order cumulants. The second important reason for closure is the following. The important terms in the regeneration of the Nth order cumulant are not the terms containing cumulants of order higher then N, but rather those terms which are products of lower order cumulants. Important means that, even though the nonlinear coupling is weak, the effects of these terms persist over long times because of resonant interaction. Furthermore, the regeneration process takes place on a much longer time scale than does the initial decorrelation process due to wave dispersion. On this long time scale, namely, the time

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taken for triad or quartet (or, as in some rare cases, quintic) resonances to produce order 1 effects, the system of equations for the cumulant hierarchy becomes closed. If ϵ is a typical dimensionless wave amplitude (for acoustic waves it is $\delta\rho/\rho_0$, the ratio of average fluctuation density amplitude to the ambient value), then this time (measured in units of the timescale ω_{I}^{-1}) is ϵ^{-2} for triad resonances and ϵ^{-4} for quartet resonances, although there is an additional frequency correction in the latter case that comes in on the ϵ^{-2} time scale.

Mathematically, these results are obtained by perturbation theory, in which the terms leading to long time cumulative effects can be identified, tabulated, and summed. The method closely parallels that of the Dyson-Wyld diagrammatic approach which will be discussed in Sec. IV. A key part of the analysis is the asymptotic ($\lim_{t\to\infty}$) evaluation of certain integrals such as

$$\int f(\mathbf{k}_r)\Delta\left[\sum_{r=1}^{N} s_r\omega(\mathbf{k}_r)\right]\delta\left(\sum_{r=1}^{N}\mathbf{k}_r\right)\Pi d\mathbf{k}_r, \tag{1.3}$$

where

$$\Delta(h)=\int_0^t dt\,\exp(iht)=\frac{\exp(iht)-1}{ih}, \tag{1.4}$$

and $\delta(x)$ is the Dirac delta function. The function $\Delta(h)$ contains the fast (oscillations of the order of ω_I^{-1}) time t, whereas the other functions in the integrand, here denoted by $f(\mathbf{k}_r)$, only change over much longer times. The exponent of $\Delta(h)$ is $\Sigma_1^N s_r\omega(\mathbf{k}_r)$ where $\omega(\mathbf{k}_r)$ is the linear dispersion relation and s_r (often $s_r=\pm 1$) denotes its multiplicity. For example, in acoustic waves, a wave vector $\mathbf{k}$ has two frequencies corresponding to waves running parallel and antiparallel to $\mathbf{k}$. The maximum contribution to integrals such as Eq. (1.3) in the limit of large time t occurs on the so called resonant manifold M, where

$$\sum_{r=1}^{N}\mathbf{k}_r=0,\quad h=\sum_{r=1}^{N} s_r\omega(\mathbf{k}_r)=0 \tag{1.5}$$

for some choices of the sequence s_r. However, the precise form of the asymptotic limit also depends on whether the zeros of h on M are simple or of higher order. For the case of (fully) dispersive waves, such as gravity waves on deep water, Rossby waves, waves of diffraction on optical beams, the zero of h is simple, and (for sufficiently smooth f) one has

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} f(h)\frac{\exp(iht)-1}{ih}dh = \pi\,\mathrm{sgn}(t)f(0)+iP\int_{-\infty}^{\infty}\frac{f(h)}{h}dh, \tag{1.6}$$

or, schematically,

$$\Delta(h)\propto\pi\,\mathrm{sgn}(t)\,\delta(h)+iP\left(\frac{1}{h}\right), \tag{1.7}$$

where P denotes Cauchy principal value. In these cases, the integrand in the kinetic equation, the equation describing the resonant transfer of spectral density, contains products of energy densities and the Dirac delta functions $\delta(\Sigma_1^N s_r\omega(\mathbf{k}_r))$ and $\delta(\Sigma_1^N\mathbf{k}_r)$, clearly indicating that spectral energy transfer takes place on the resonant manifold M. The asymptotic equations for the change of the higher order cumulants can be interpreted as a complex frequency modification whose real part describes the expected nonlinear shift in frequency, and whose imaginary part describes a broadening of the resonant manifold along its normal directions.

But acoustic waves are not fully dispersive. The linear dispersion relation

$$\omega(\mathbf{k})=c|k|=c\sqrt{k_\parallel^2+k_\perp^2},\quad \mathbf{k}=(k_\parallel,k_\perp) \tag{1.8}$$

where c is the sound speed, leads to a dispersion tensor which has rank $(d-1)$. As we will see, this changes the asymptotic. Furthermore, three wave resonances occur between wave vectors which are purely collinear. Therefore, since the kinetic equation (KE) only considers wave interaction on the resonant manifold, there is no way of redistributing energy out of a given direction. At best, the KE will only describe spectral energy transfer along rays in wave-vector space. Moreover, depending on dimension d, the long time behavior of integrals (1.3) differ greatly. For a given vector $\mathbf{k}$, the locus of the resonant partners $\mathbf{k}_1$ and $\mathbf{k}-\mathbf{k}_1$ in a resonant triad is given by the surface in $\mathbf{k}_1$ space defined by

$$h(\mathbf{k}_1)=s_1k_1+s_2|k-k_1|-s|k|=0. \tag{1.9}$$

Here $s,s_1,s_2=\pm 1$. For $d=1$ and the appropriate choices of the wave directions s_1, s_2, and s, this manifold is *all* $\mathbf{k}_1$. Therefore the fast oscillations in the integral are of no consequence, and do not cause any decorrelation to occur. All waves moving in the same directions travel with the same speed. Initial correlations are completely preserved. Moreover, we know that for one-dimensional compressible flow, nonlinear terms, no matter how weak initially, eventually lead to finite time multivalued solutions. Assuming the usual viscous regularization, multivalued solutions are replaced by shocks, namely, almost discontinuous solutions where discontinuities are resolved across very thin viscous layers. One would naturally expect an energy spectrum $E_1(k)$ which reflects this fact, namely,

$$E_1(k)\propto 1/k^2. \tag{1.10}$$

In two dimensions, one has dispersion (diffraction) in one direction. Indeed, for $d>1$, while

$$\nabla_{\mathbf{k}_1}h=0 \tag{1.11}$$

on the manifold M, the Hessian of $h(\mathbf{k}_1)$ is not identically zero. In two dimensions, integral (1.3) behaves as

$$\int f(x)\frac{\exp(ix^2t)-1}{ix^2}dx\propto 2t\int f(x)\exp(ix^2t)dx, \tag{1.12}$$

which grows as $t^{1/2}$ as $t\to\infty$. In three dimensions, the growth is much weaker. Since that is the case we will look at in detail, we give the exact result. Let

$$\mathbf{k}=(K>0,0,0),\quad \mathbf{k}_1=(K_x,K_y,K_z),$$
$$\mathbf{k}_2=(K-K_x,-K_y,-K_z).$$

Then, for $s_1=s_2=s$,

$$h=c(s_1|\mathbf{k}_1|+s_2|\mathbf{k}-\mathbf{k}_1|-sK)$$
$$=\frac{sKc}{2K_x(K-K_x)}(K_y^2+K_z^2)+O(K_y^3,K_y^2K_z,\dots) \tag{1.13}$$

near the resonant value $(K,0,0)$. The integral

$$\int_{-\infty}^{\infty} f(K_x,K_y,K_z;s_1,s_2)\frac{e^{iht}-1}{ih}dK_xdK_ydK_z$$

tends to

$$\alpha\int_{-\infty}^{0} f(K_x,0,0;-s,s)(-K_x)(K-K_x)dK_x$$
$$+\alpha\int_{0}^{K} f(K_x,0,0;s,s)K_x(K-K_x)dK_x$$
$$+\alpha\int_{K}^{\infty} f(K_x,0,0;s,-s)K_x(K_x-K)dK_x$$
$$-\frac{2i\alpha s}{\pi}\ln t\int_{-\infty}^{0} f(K_x,0,0;-s,s)K_x(K-K_x)dK_x$$
$$-\frac{2i\alpha s}{\pi}\ln t\int_{0}^{K} f(K_x,0,0;s,s)K_x(K-K_x)dK_x$$
$$-\frac{2i\alpha s}{\pi}\ln t\int_{K}^{\infty} f(K_x,0,0;s,-s)K_x(K-K_x)dK_x \tag{1.14}$$

in the limit $t\to\infty$. Here $\alpha=\pi^2/Kc$ and we have kept only the leading order real and imaginary contributions. The essential difference from Eqs. (1.6) and (1.7) is the additional Dirac delta function multiplied by $\ln t$ in the imaginary term. This will not change the kinetic equation for the spectral energy density. If we write the total energy per unit volume E as

$$E=2\rho_0c^2\epsilon^2\int e(\mathbf{k})d\mathbf{k}, \tag{1.15}$$

where ρ_0 is the ambient density and ϵ a measure of amplitude, then

$$\frac{de(\mathbf{k})}{dt}=St(e,\dot{e})$$

$$St(e,\dot{e})=\frac{\pi^2c(\mu+1)^2\epsilon^2K^4}{4}\Bigg\{2\int_0^\infty d\gamma\gamma(\gamma+1)$$
$$\times[e(\gamma\mathbf{k})e((\gamma+1)\mathbf{k})+\gamma e(\mathbf{k})e((\gamma+1)\mathbf{k})$$
$$-(\gamma+1)e(\mathbf{k})e(\gamma\mathbf{k})]$$
$$+\int_0^1 dk\alpha(1-\alpha)[e(\alpha k)e((1-\alpha)\mathbf{k})$$
$$-\alpha e(\mathbf{k})e((1-\alpha)\mathbf{k})-(1-\alpha)e(k)e(\alpha\mathbf{k})]\Bigg\}, \tag{1.16}$$

where μ is the adiabatic constant $[p=p_0(\rho/\rho_0)^\mu]$ and $|\mathbf{k}|=K$. In d dimensions a little calculation show, that the right-hand side of Eq. (1.16) has the t dependence $t^{(3-d/2)}$, so that in general the nonlinear interaction time τ_{NL} for the resonant exchange of spectral energy is $\epsilon^2t^{(5-d)/2}=O(1)$ or $\tau_{NL}\propto\epsilon^{-4/(5-d)}$. (Note that for $d\geq5$, there is no cumulative effect of this resonance.)

While the extra term in Eq. (1.14) proportional to $i\ln t$ plays no role in the spectral energy transfer, it will, however, appear in the frequency modification. Calculating the long time behavior of the higher order cumulants leads to a natural renormalization of the frequency,

$$\omega(\mathbf{k})=c|\mathbf{k}|\left[1-2\pi(\mu+1)^2\epsilon^2\ln\frac{1}{\epsilon^2}\int_0^\infty\beta^2e(\beta\hat{k})d\beta+O(\epsilon^2)\right]$$
$$+i\pi^2(\mu+1)^2\epsilon^2\Bigg[\int_{|\mathbf{k}|}^\infty\beta^2e(\beta\hat{k})d\beta$$
$$+\frac{1}{|\mathbf{k}|}\int_0^{|\mathbf{k}|}\beta^3e(\beta\hat{k})d\beta+|k|\int_0^{|\mathbf{k}|}\beta e(\beta\hat{k})d\beta\Bigg], \tag{1.17}$$

where $\hat{k}=\mathbf{k}/K$. The calculation of the frequency renormalization is the main result of this paper. We present two derivations of this result, in the framework of the above analysis and making use of a diagrammatic perturbation approach.

Equation (1.16) is nothing but a "regular" kinetic equation for the three-wave interactions, written in a dispersionless limit $\omega=c|\mathbf{k}|$. In this case three-wave resonant conditions

$$\pm\omega(\mathbf{k})=\pm\omega(\mathbf{k}_1)\pm\omega(\mathbf{k}_2),\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \tag{1.18}$$

can be satisfied if and only if all three vectors $\mathbf{k}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ are parallel; as a result, the integration over $\mathbf{k}_1$, and $\mathbf{k}_2$ is along the line parallel to $\mathbf{k}$. It is unclear *a priori* that the three-wave kinetic equation can be used in the dispersionless case; is certainly less plausible in the two-dimensional case, where the formal implementation of the kinetic equation leads to stronger divergences.

The derivation presented above is taken from the article of Newell and Aucoin [9], who made the first serious attempt at an analytical description of the dispersionless acoustic turbulence. Newell and Aucoin [9] also argued that a natural asymptotic closure is also obtained in two dimensions because of the relative higher asymptotic growth rates of terms in the kinetic equation involving only the spectral energy. However this is still a point of dispute, it is not yet resolved and will not be addressed further here.

Independently the kinetic equation (1.16) was applied to acoustic turbulence by Zakharov and Sagdeev [8], who used it just as a plausible hypothesis. However, Zakharov and Sagdeev also suggested an explicit expression for the spectrum of acoustic turbulence

$$e(k)\propto k^{-3/2}, \tag{1.19}$$

which is just a Kolmogorov-type spectrum, first obtained by Kolmogorov from dimensional considerations in the context of hydrodynamic turbulence. Here, however, Eq. (1.19) is an exact solution of the equation

$$St(e,\dot{e})=0. \tag{1.20}$$

The proof of this fact can be found in Ref. [1]. One should also mention that the quantum kinetic equation was applied to a description of a system of weakly interacting dispersionless phonons as long ago as 1937 by Landau and Rumer [10].

Kadomtsev and Petviashvili [11] criticized this result on the grounds that the kinetic equation in the dispersionless case can hardly be justified because of the special nature of the linear dispersion relation. They suggested that acoustic turbulence in two and three dimensions was much more likely to have parallels with its analog in one dimension. We have already mentioned in that case that the usual statistical description is inadequate both because there is no decorrelation dynamics and because shocks form no matter how weak the nonlinearity initially is. The equilibrium statistics relevant in that case is much more likely to be a random distribution of discontinuities in the density and velocity fields which lead to an energy distribution of Eq. (1.10). Further, Kadomtsev and Petviashvili argued that even in two and three dimensions one would expect the same result, namely,

$$k^{d-1}e(\mathbf{k})\propto k^{-2}, \tag{1.21}$$

a random distribution of statistically independent shocks propagating in all directions.

However, wave packets traveling in almost parallel directions are not independent. Consider a solid angle containing $N=(k_\parallel/k_\perp)^{d-1}$ wave packets with wave vectors $(k_\parallel,k_\perp)$, where $k_\parallel=l^{-1}$ is a typical length scale of the fluctuating field in the direction of the propagation, and $k_\perp\ll k_\parallel$. The shock time $\tau_{\rm sh}$ for a single wave packet would be $l\sqrt{\rho N/E}$ $\propto(l/c\epsilon)N^{(1/2)}$, where E is the total energy in the field. The dispersion (diffraction) time $\tau_{\rm disp}$, namely, the time over which several different packets have time to interact linearly, is of the order of $k_\parallel/(ck_\perp^2)\propto lN^{2/(d-1)}/c$. As we have already observed, the nonlinear resonance interaction time $\tau_{\rm NL}$ for spectral energy transfer is $(l/c)\epsilon^{-(4/(5-d))}$. The ration is $\tau_{\rm disp}:\tau_{\rm sh}:\tau_{\rm NL}=N^{2/(d-1)}:N^{1/2}\epsilon^{-1}:\epsilon^{-4/(5-d)}$. In the limits $N\to\infty$ and $\epsilon\to 0$, the shock time is sandwiched between the linear dispersion time and nonlinear interaction time, and, if we choose $N(\epsilon)$ by equating the first two, all three are the same. Moreover, the phase mixing, which occurs due to the crossing of acoustic wave beams, occurs on a shorter time scale, a fact that suggests that the resonant exchange of energy is the more important process. But even then, several very important questions remain.

(1) To what distribution does the energy along a given wave-vector ray relax?

(2) How does energy become shared between neighboring rays?

(3) Does energy tend to diffuse away from the ray with maximum energy, or can it focus onto that ray? In the latter case, one might argue that shock formation may again become the relevant process, especially if the energy should condense on rays with very different directions.

The aim of this paper is to take a very modest first step in the direction of answering these questions. In particular, we present a curious result. The fact that there is a strong $(\epsilon^2\ln 1/\epsilon^2)$ correction to the frequency leads us to ask if that terms could provide the dispersion required to allow the usual triad resonance process to carry energy between neighboring rays. At first sight, it would appear that that is indeed the case, that the modified nonlinear dispersion law is

$$\omega(\mathbf{k})=c(\mathbf{k})\left(1+\epsilon^2\ln\frac{1}{\epsilon^2}\Omega(k)\right), \tag{1.22}$$

where Ω is proportional to $|\mathbf{k}|$. But a surprising and nontrivial cancellation occurs, which means that the first corrections to the wave speed still keeps the system nondispersive in the propagation direction.

While this fact is the principal result of this paper, our approach lays the foundation for a systematic evaluation of the contribution to energy exchange that occurs at higher order. Indeed, we expect that some of the terms found by Benney and Newell [5], involving gradients across resonant manifolds which, in the fully dispersive case, are not relevant because the resonant three-wave interaction gives rise to an isotropic distribution, may be more important in this context.

The paper is written as follows. In Sec. II, we derive the equation of motion for acoustic waves of small but finite amplitude. A second approach discussed in Sec. II B starts from the Hamiltonian formulation of the Euler equations, and again makes use of the small amplitude parameter of the problem to simplify the interaction Hamiltonian. As we will see in Sec. II C, both approaches are equivalent and which approach to use is the question of taste.

Next, in Sec. III we write down the hierarchy of equations for the spectral cumulants and solve them perturbatively. Certain resonances manifest themselves as algebraic and logarithmic time growth in the formal perturbation expansions, and mean that these expansions are not uniformly asymptotic in time. The kinetic equation, describing the long time behavior of the zeroth order spectral energy, and the equations describing the long time behavior of the zeroth order higher cumulants, are simply conditions that effectively sum the effect of the unbounded growth terms. Under this renormalization, the perturbation series becomes asymptotically uniform. By asymptotically uniform, we mean that the asymptotic expansion for each of the cumulants remains an asymptotic expansion over long times. All unbounded growths are removed. While this procedure in principle requires one to identify and calculate unbounded terms to all orders, in practice one gains a very good approximation by demanding uniform asymptotic behavior only to that order in the coupling coefficient where the unboundedness first appears.

In other words, this means that if one finds that if the first two terms of the asymptotic expansion are $1+\epsilon^2 t\psi_1+\cdots$, then the effective removal of ψ_1 will remove all terms which are powers of $(\epsilon^2 t)$ in the full expansion. Likewise, it also assumes that there appear no worse secular terms at a higher order, such as, for example, $\epsilon^4 t^3\psi_2$. To achieve uniformity, one requires an intimate knowledge of how unbounded growth appears. This sort of perturbative analysis was first done in the 1930s by Dyson. A technical innovation was to use graph notations, called *diagrams*, for representing lengthy analytical expressions for high order terms in the perturbation series. It often happens that one can find the principal subsequence of terms just by looking on the topological structure of corresponding diagrams. This method of treating perturbation approaches is called *the diagrammatic technique*.

The first variant of diagrammatic technique for nonequilibrium processes was suggested by Wyld [12] in the context of the Navier-Stokes equation for an incompressible fluid. This technique was later generalized by Martin, Siggia, and Rose [13], who demonstrated that it may be used to investigate the fluctuation effects in the low-frequency dynamics of any condensed matter system. In fact this technique is also a classical limit of the Keldysh diagrammatic technique [14] which is applicable to any physical system described by interacting Fermi and Bose fields. Zakharov and L'vov [15] extended the Wyld technique to the statistical description of Hamiltonian nonlinear-wave fields, including hydrodynamic turbulence in the Clebsch variables [16]. In Sec. IV, we will use this particular method for treating acoustic turbulence.

Note that in such a formulation, unbounded growths appear as divergences (or almost divergences) due to the presence of zero denominators caused by resonances—the very same resonances, in fact, that give rise to unbounded growth in our more straightforward perturbation approach. Moreover, diagrammatic techniques are schematic methods for identifying all problem terms and for adding them up. If one uses the diagram technique only to the first order at which the first divergences appear, this is called the one-loop approximation, and is equivalent to identifying the first long time nonlinear effects. This is exactly analogous to what we will do in our first approach in this paper, although we will also display the diagram technique. The one loop approximation will give the same long time behavior of the system for times of $\tau_{\rm NL}$ defined earlier. In Appendix C we analyze two-loop diagrams, and show that some of them gives the same order contribution to γ_k as two-loop diagrams. Nevertheless one may believe that even the one-loop approximation gives a qualitatively correct description of the dynamics of the system.

Section V is devoted to some concluding remarks and the identification of the remaining challenges. We now begin with deriving the basic equations of motion for weak acoustic turbulence.

II. BASIC EQUATION OF MOTION FOR WEAK ACOUSTIC TURBULENCE

A. Straightforward approach

Consider the Euler equations for a compressible fluid:

$$\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0, \tag{2.1}$$

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p(\rho)/\rho.$$

Here $v(\mathbf{x},t)$ is the Euler fluid velocity, $\rho(\mathbf{x},t)$ the density, and $p(\mathbf{r},t)$ the pressure which, in the general case, is a function of fluid density and specific entropy s [$p=p(\rho,s)$]. In ideal fluids where there is no viscosity and heat exchange, the entropy per unit volume is carried by the fluid, i.e., it obeys the equation $\partial s/\partial t+(\mathbf{v}\cdot\nabla)s=0$. A fluid in which the specific entropy is constant throughout the volume is called barotropic; the pressure in such a fluid is a single-valued function of the density $p=p(\rho)$. In this case, $\nabla p/\rho$ may be expressed via the gradient of specific enthalpy of unit mass $w=E+pV$ and $dw=Vdp=dp/\rho$. Thus $\nabla p/\rho=\nabla w$.

Writing the fluid density $\rho(\mathbf{x},t)$ as $\rho_0(1+\eta(\mathbf{x},t))$, the velocity field as $v(\mathbf{x},t)$, the pressure field as $p=p_0(1+\eta)^\mu$, and the enthalpy as

$$w=\int\frac{dp}{\rho}=\frac{c_0^2}{\mu-1}\left(1+(\mu-1)\eta+\frac{(\mu-1)(\mu-2)}{2}\eta^2+\cdots\right),$$

one can write Eq. (2.1) to third order in amplitude in the following forms:

$$\frac{\partial\eta}{\partial t}+\frac{\partial v_i}{\partial x_i}=-\frac{\partial}{\partial x_i}\eta v_i, \tag{2.2}$$

$$\frac{\partial v_j}{\partial t}+c^2\frac{\partial\eta}{\partial x}=-v_m\frac{\partial v_i}{\partial x_m}-\frac{c^2(\mu-2)}{2}\frac{\partial}{\partial x_j}\eta^2-\frac{c^2(\mu-2)(\mu-3)}{6}\frac{\partial}{\partial x_j}\eta^3. \tag{2.3}$$

Let us introduce new variables as

$$\eta(\mathbf{x},t)=\int\sum_s\epsilon a^s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}+is\omega(\mathbf{k})t}d\mathbf{k}, \tag{2.4}$$

$$v_j(\mathbf{x},t)=\int\sum_s\frac{-c^2k_j}{s\omega(\mathbf{k})}\epsilon a^s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}+is\omega(\mathbf{k})t}d\mathbf{k}, \tag{2.5}$$

where $0<\epsilon\ll1$, $\omega(\vec{k})=c|\mathbf{k}|$ and Σ_s connotes summation over $s=\pm1$. From Eqs. (2.2) and (2.3),

$$\begin{aligned}\frac{\partial a^s(\mathbf{k},t)}{\partial t}=&\epsilon\sum_{s_p,s_q}\int d\mathbf{k}_p d\mathbf{k}_q L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}a^{s_p}(\mathbf{k}_p,t)a^{s_q}(\mathbf{k}_q,t)\\&\times\delta(\mathbf{k}_p+\mathbf{k}_q-\mathbf{k})\\&\times\exp\{i[s_p\omega(\mathbf{k}_p)+s_q\omega(\mathbf{k}_q)-s\omega(\mathbf{k})]t\}\\&+\epsilon^2\sum_{s_p,s_q,s_r}\int d\mathbf{k}_p d\mathbf{k}_q d\mathbf{k}_r L^{s,s_p,s_q,s_r}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q,\mathbf{k}_r}a^{s_p}\\&\times(\mathbf{k}_p,t)a^{s_q}(\mathbf{k}_q,t)a^{s_r}(\mathbf{k}_r,t)\delta(\mathbf{k}_p+\mathbf{k}_q+\mathbf{k}_r-\mathbf{k})\\&\times\exp\{i[s_p\omega(\mathbf{k}_p)+s_q\omega(\mathbf{k}_q)+s_r\omega(\mathbf{k}_r)\\&-s\omega(\mathbf{k})]t\},\end{aligned} \tag{2.6}$$

where the summation is done over all signs of s_p, s_q, and s_r, and we use the shorthand notation $\omega_p=\omega(\mathbf{k}_p)$. The coupling coefficients are

$$L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}=\frac{ic^2}{4}\left(\frac{\mathbf{k}\cdot\mathbf{k}_p}{s_p\omega_p}+\frac{\mathbf{k}\cdot\mathbf{k}_q}{s_q\omega_q}+\frac{s\omega}{s_p\omega_p s_q\omega_q}\mathbf{k}_p\cdot\mathbf{k}_q\right)+\frac{i}{4}(\mu-2)s\omega \tag{2.7}$$

$$L^{s,s_p,s_q,s_r}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q,\mathbf{k}_r}=\frac{i\omega}{12}(\mu-2)(\mu-3). \tag{2.8}$$

These coefficients have the following important properties:

(i) $L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}$ is symmetric under the interchange of p and q.

(ii) $L^{s_p,s,-s_q}_{\mathbf{k}_p,\mathbf{k},-\mathbf{k}_q}=(s_p\omega_p/s\omega)L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}$.

(iii) On the resonant manifold M, given by

$$\frac{1}{c}h=s_p|\mathbf{k}_p|+s_q|\mathbf{k}-\mathbf{k}_p|-s|\mathbf{k}|=0, \tag{2.9}$$

$$L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}=\frac{ics}{4}(\mu+1)K, \tag{2.10}$$

where $|\mathbf{k}|=K$. Note that if $\mathbf{k}=(K,0,0)$, the resonant manifold is not of codimension 1, but degenerates to $K_y=K_z=0$, where $\mathbf{k}_p=(K_x,K_y,K_z)$, $\mathbf{k}_q=(K-K_x,-K_y,-K_z)$. There are three cases. (1) For $K_x<0<K$, $|\mathbf{k}_p|=-K_x$, $|\mathbf{k}_q|=K+K_x$, $s_p=-s$, $s_q=s$. (2) For $0<K_x<K$, $|\mathbf{k}_p|=K_x$, $|\mathbf{k}_q|=K-K_x$, $s_p=s_q=s$. (3) For $0<K<K_x$, $|\mathbf{k}_p|=K_x$, $|\mathbf{k}_q|=K_x-K$, $s_p=s, s_q=-s$.

B. Hamiltonian description of acoustic turbulence

1. Equations of motion and canonical variables

Consider again the Euler equations for a compressible fluid, Eqs. (2.1). The enthalpy of a unit mass $w=E+pV$ is equal to the derivative of internal energy of unit volume $\varepsilon(\rho)=E\rho$ with respect to fluid density: $w=\delta\varepsilon/\delta\rho$. As a result of direct differentiation with respect to time, it is readily evident that equations (2.1) conserve the energy of the fluid

$$\mathcal{H}=\int[\rho v^2/2+\varepsilon(\rho)]\,d\mathbf{r}. \tag{2.11}$$

One can show (see, for example, [1]) that Eqs. (2.1) may be written in the Hamiltonian forms

$$\partial\rho/\partial t=\delta\mathcal{H}/\delta\Phi,\quad \partial\Phi/\partial t=-\delta\mathcal{H}/\delta\rho, \tag{2.12}$$

$$\partial\lambda/\partial t=\delta\mathcal{H}/\delta\mu,\quad \partial\mu/\partial t=-\delta\mathcal{H}/\delta\lambda, \tag{2.13}$$

if the velocity $\mathbf{v}(\mathbf{r},t)$ is presented in terms of two pairs of Clebsch variables (ρ,Φ) and (λ,ν) as follows:

$$\mathbf{v}=\lambda\frac{\nabla\mu}{\rho}+\nabla\Phi. \tag{2.14}$$

Here the energy (2.11) is expressed in terms (ρ,Φ) and (λ,ν), so that Eq. (2.14) becomes the Hamiltonian of the system. As seen from Eq. (2.14), the case with $\lambda=0$ or $\mu=\text{const}$ corresponds to potential fluid motions which are defined by a pair of variables (ρ,Φ) according to Eqs. (2.12). It is convenient to transform in the $\mathbf{k}$ representation from the real canonical variables, $\Phi(\mathbf{k}),\rho(\mathbf{k})$ to the complex ones $b(\mathbf{k})$ and $b^*(\mathbf{k})$,

$$\Phi(\mathbf{k})=-i\sqrt{(c/2\rho_0 k)}[b(\mathbf{k})-b^*(-\mathbf{k})], \tag{2.15}$$

$$\delta\rho(\mathbf{k})=\sqrt{(\rho_0 k/2)}[b(\mathbf{k})+b^*(-\mathbf{k})]. \tag{2.16}$$

Here $\delta\rho(\mathbf{k})=[\rho(\mathbf{k})-\rho_0(\mathbf{k})]$ is the Fourier transform of density deviation from the steady state.

2. Hamiltonian of acoustic turbulence

Let us expand the Hamiltonian (2.11) (expressed in terms of b, b^*) in power series

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\text{int}}. \tag{2.17}$$

Here $\mathcal{H}_0$ is quadratic in b and b^*, giving the Hamiltonian of noninteracting waves:

$$\mathcal{H}_0=\int ckb(\mathbf{k})b^*(\mathbf{k})d\mathbf{k}, \tag{2.18}$$

with linear dispersion relation $\omega_0(\mathbf{k})=ck$. In the Hamiltonian of interaction $\mathcal{H}_{\text{int}}$, we take into account only three-wave processes:

$$\mathcal{H}_{\text{int}}=\tfrac{1}{2}\int(V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)b_1^*b_2b_3+\text{c.c.})\times\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3. \tag{2.19}$$

Here we neglected $0\leftrightarrow3$ processes (processes described by $b_1^*b_2^*b_3^*$ and $b_1b_2b_3$ terms), because they are nonresonant. This means that if we take into account the $0\leftrightarrow3$ term, it is not going to change our final results; thus we can neglect it from the very beginning. We also neglected contributions from four-wave and higher terms, because three-wave interaction is the dominant one.

The coupling coefficient of the three-wave interaction a given by [1]

$$V(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2)=\left(\frac{ckk_1k_2}{4\pi^3\rho_0}\right)^{1/2}(3g+\cos\theta_{01}+\cos\theta_{02}+\cos\theta_{12}), \tag{2.20}$$

where g is some dimensionless constant of the order of unity, and θ_{ij} is the angle between $\mathbf{k}_i$ and $\mathbf{k}_j$. Since we have an almost linear dispersion relation, only almost parallel wave vectors can interact; therefore $\cos\theta_{ij}$ with high accuracy can be replaced by 1, and Eq. (2.20) reduces to

$$V(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2)=\left(\frac{ckk_1k_2}{4\pi^3\rho_0}\right)^{1/2}3(g+1). \tag{2.21}$$

3. Canonical equation of motion

The Hamiltonian equations of motion (2.12) for the complex canonical variables b and b^* have the standard form [1]

$$i\frac{\partial b(\mathbf{k},t)}{\partial t}=\frac{\delta\mathcal{H}}{\delta b^*(\mathbf{k},t)}. \tag{2.22}$$

For the acoustic Hamiltonians (2.17)–(2.19), this equation takes the form

$$\left[i\frac{\partial}{\partial t}-ck\right]b(\mathbf{k},t)=\tfrac{1}{2}\int V(\mathbf{k},\mathbf{q},\mathbf{p})b(\mathbf{q})b(\mathbf{p}) \times\delta(\mathbf{k}-\mathbf{q}-\mathbf{p})\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^3} +\int V^*(\mathbf{k},\mathbf{q},\mathbf{p})b(\mathbf{q})^*b(\mathbf{p}) \times\delta(\mathbf{k}+\mathbf{q}-\mathbf{p})\frac{d^3\mathbf{q}\,d^3\mathbf{p}}{(2\pi)^3}. \tag{2.23}$$

It is sometimes convenient to concentrate attention on steady state turbulence, which may be described in the $\mathbf{k},\omega$ representation. After performing a time Fourier transform, one has, instead of (2.23),

$$[\omega-ck]b(\mathbf{k},\omega)=\tfrac{1}{2}\int V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)b_1b_2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \times\delta(\omega-\omega_1-\omega_2)\frac{d\mathbf{k}_1d\omega_1d\mathbf{k}_2d\omega_2}{(2\pi)^4} +\int V^*(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)b_1^*b_2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) \times\delta(\omega+\omega_1-\omega_2)\frac{d\mathbf{k}_1d\omega_1d\mathbf{k}_2d\omega_2}{(2\pi)^4}\ . \tag{2.24}$$

Hereafter we will refer to this as the *basic equation of motion for the acoustic turbulence normal variables* b_k, *and* b_k^*, and use it for a statistical description of acoustic turbulence.

C. Relations between wave amplitudes $a^+(\mathbf{k})$, $a^-(\mathbf{k})$ with normal variables of acoustic turbulence $b(\mathbf{k})$, $b^*(\mathbf{k})$

Comparing Eqs. (2.4) and (2.5), we obtain

$$\delta\rho(\mathbf{k},t)=\rho_0\epsilon\{a^+(\mathbf{k},t)\exp[i\omega(\mathbf{k})t] +a^-(\mathbf{k},t)\exp[-i\omega(\mathbf{k})t]\}(2\pi)^{3/2}, \tag{2.25}$$

$$\Phi(\mathbf{k},t)=\frac{ic^2\epsilon}{\omega(\mathbf{k})}\{a^+(\mathbf{k},t)\exp[i\omega(\mathbf{k})t] -a^-(\mathbf{k},t)\exp[-i\omega(\mathbf{k})t]\}(2\pi)^{3/2}, \tag{2.26}$$

Here Φ is velocity potential: $\mathbf{v}=\nabla\Phi$. This gives

$$a^+(\mathbf{k},t)=\frac{\exp[-i\omega(\mathbf{k})t]}{2\epsilon(2\pi)^{3/2}}\left[\frac{\delta\rho(\mathbf{k},t)}{\rho_0}-i\Phi(\mathbf{k},t)\frac{\omega(\mathbf{k})}{c^2}\right],$$

$$a^-(\mathbf{k},t)=\frac{\exp[i\omega(\mathbf{k})t]}{2\epsilon(2\pi)^{3/2}}\left[\frac{\delta\rho(\mathbf{k},t)}{\rho_0}+i\Phi(\mathbf{k},t)\frac{\omega_k}{c^2}\right]. \tag{2.27}$$

Note that a^+ and a^- are dimensionless variables.

Now we can easily express $a^+(\mathbf{k})$ and $a(\mathbf{k})$ in terms of $b(\mathbf{k})$ and $b^*(\mathbf{k})$, and thereby relate the two alternative approaches presented in this paper,

$$a^+(\mathbf{k},t)=\frac{1}{\epsilon}\left(\frac{k}{2c\rho_0}\right)^{1/2}(2\pi)^{-3/2}\exp[-i\omega(\mathbf{k})t]b^*(-\mathbf{k}), \tag{2.28}$$

$$a^-(\mathbf{k},t)=\frac{1}{\epsilon}\left(\frac{k}{2c\rho_0}\right)^{1/2}(2\pi)^{-3/2}\exp[i\omega(\mathbf{k})t]b(\mathbf{k}). \tag{2.29}$$

To check that the two approaches are consistent, we rewrite the equation of motion (2.6) for a_k^s, neglecting ϵ^2 (four-wave interaction) terms,

$$\frac{\partial a^s(\mathbf{k},t)}{\partial t}=\epsilon\sum_{s_ps_q}\int d\mathbf{k}_pd\mathbf{k}_qL^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}a^{s_p}(\mathbf{k}_p,t)a^{s_q}(\mathbf{k}_q,t) \times\delta(\mathbf{k}_p+\mathbf{k}_q-\mathbf{k}) \times\exp\{i[s_p\omega(\mathbf{k}_p)+s_q\omega(\mathbf{k}_q)-s\omega(\mathbf{k})]t\}. \tag{2.30}$$

Now we substitute Eqs. (2.28) and (2.29) into Eq. (2.30), and obtain

$$\left[\frac{\partial}{\partial t}+i\omega(\mathbf{k})\right]b(\mathbf{k},t)=-i\int d\mathbf{p}d\mathbf{q}\left(\frac{kpqc}{4\pi^3\rho_0}\right)^{1/2} \times[(\mu-2)+\cos\theta_{\mathbf{k},\mathbf{p}}+\cos\theta_{\mathbf{k},\mathbf{q}} +\cos\theta_{\mathbf{p},\mathbf{q}}] \times[\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})b_{\mathbf{p}}^*b_{\mathbf{q}}^*+2\delta_{\mathbf{k}+\mathbf{p}-\mathbf{q}}b_{\mathbf{p}}^*b_{\mathbf{q}} +\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})b_{\mathbf{p}}b_{\mathbf{q}}]. \tag{2.31}$$

Now one can see that Eq. (2.31) looks exactly like Eq. (2.22), with Hamiltonian (2.19) and coupling coefficient (2.20). Thus one concludes that the two approaches are equivalent, and that a choice between them is a question of taste.

III. LONG-TIME ANALYSIS OF STATISTICAL BEHAVIOR

The analysis proceeds by first forming the hierarchy of equations for the spectral cumulants (correlation functions of the wave amplitudes) defined as follows. The mean is zero.

$$\langle a^s(\mathbf{k})a^{s'}(\mathbf{k}')\rangle=\delta(\mathbf{k}+\mathbf{k}')q^{ss'}(\mathbf{k},\mathbf{k}'), \tag{3.1}$$

$$\langle a^s(\mathbf{k})a^{s'}(k')a^{s''}(\mathbf{k}'')\rangle=\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')q^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}''), \tag{3.2}$$

$$\begin{aligned}&\langle a^s(\mathbf{k})a^{s'}(\mathbf{k}')a^{s''}(\mathbf{k}'')a^{s'''}(\mathbf{k}''')\rangle\\&\quad=\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')q^{ss's''s'''}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''')\\&\qquad+\delta(\mathbf{k}+\mathbf{k}')\delta(\mathbf{k}''+\mathbf{k}''')q^{ss'}(\mathbf{k},\mathbf{k}')q^{s''s'''}(\mathbf{k}'',\mathbf{k}''')\\&\qquad+\delta(\mathbf{k}+\mathbf{k}'')\delta(\mathbf{k}'+\mathbf{k}''')q^{ss''}(\mathbf{k},\mathbf{k}'')q^{s's'''}(\mathbf{k}',\mathbf{k}''')\\&\qquad+\delta(\mathbf{k}+\mathbf{k}''')\delta(\mathbf{k}'+\mathbf{k}'')q^{ss'''}(\mathbf{k},\mathbf{k}''')q^{s's''}(\mathbf{k}',\mathbf{k}''),\end{aligned} \tag{3.3}$$

where $\langle\cdots\rangle$ denotes average and the presence of the δ function is a direct reflection of the spatial homogeneity. Indeed, the property of spatial homogeneity affords one a way of defining averages, which does not depend on the presence of a joint distribution. We can define the average $\langle\eta(\mathbf{x})\eta(\mathbf{x}+\mathbf{r})\rangle$ as simply an average over the base coordinate, namely,

$$\langle\eta(\mathbf{x})\eta(\mathbf{x}+\mathbf{r})\rangle=\frac{1}{(2L)^3}\int_{-L}^{L}\eta(\mathbf{x})\eta(\mathbf{x}+\mathbf{r})d\mathbf{x}. \tag{3.4}$$

To derive the main results of this paper, it is sufficient to write the equations for the second and third order cumulants. They are

$$\begin{aligned}\frac{dq^{ss'}_{\mathbf{k}\cdot\mathbf{k}'}}{dt}&=\epsilon P_{00'}\sum_{s_qs_p}\int d\mathbf{k}_pd\mathbf{k}_qL^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}q^{s's_ps_q}_{\mathbf{k}'\mathbf{k}_p\mathbf{k}_q}\\&\quad\times\exp[i(s_p\omega_p+s_q\omega_q-s\omega_k)t]\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}),\\&\mathbf{k}+\mathbf{k}'=0,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{dq^{ss's''}_{\mathbf{k}\cdot\mathbf{k}'\mathbf{k}''}}{dt}&=\epsilon P_{00'0''}\int d\mathbf{k}_pd\mathbf{k}_qL^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q}q^{s',s'',s_p,s_q}_{\mathbf{k}',\mathbf{k}'',\mathbf{k}_p,\mathbf{k}_q}\\&\quad\times\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\exp[i(s_p\omega_p+s_q\omega_g-s\omega_k)t]\\&\quad+2\epsilon P_{00'0''}\sum_{s_ps_q}L^{s,s_p,s_q}_{\mathbf{k},-\mathbf{k}',-\mathbf{k}''}q^{s',s_p}_{\mathbf{k}',-\mathbf{k}'}q^{s'',s_q}_{\mathbf{k}'',-\mathbf{k}''}\\&\quad\times\exp[i(s_p\omega'+s_q\omega''-s\omega)t],\\&\mathbf{k}+\mathbf{k}'+\mathbf{k}''=0,\end{aligned} \tag{3.6}$$

where the symbol $P_{00'}$ ($P_{00'0''}$) means that we sum over all replacements $0\to0'$, $0'\to0$ ($0\to0',0'\to0''$, $0''\to0$, $0\to0'',0'\to0$, $0''\to0'$).

The total energy of the system per unit volume can be written as

$$\begin{aligned}&\lim_{r\to0}\Big\langle\rho_0v_j(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})+\frac{c^2\rho_0}{\mu}\eta(\mathbf{x})\eta(\mathbf{x}+\mathbf{r})\\&\quad+\frac{\rho c^2}{2\mu}(\mu-2)\eta(\mathbf{x})\eta(\mathbf{x}+\mathbf{r})\Big\rangle\\&\quad=\lim_{\mathbf{r}\to0}\sum_{s_1s_2}\int\frac{\rho_0c^2\epsilon^2}{2}(1-s_1s_2)q^{s_1s_2}(\mathbf{k})e^{i\mathbf{kr}}d\mathbf{k}\\&\quad=\lim_{\mathbf{r}\to0}\int\rho_0c^2\epsilon^2(q^{+-}(\mathbf{k})+q^{-+}(\mathbf{k}))e^{i\mathbf{kr}}d\mathbf{k}\\&\quad=\int2\rho_0c^2\epsilon^2q^{+-}(\mathbf{k})d\mathbf{k},\end{aligned} \tag{3.7}$$

since $q^{+-}(\mathbf{k})=q^{-+}(-\mathbf{k})$. The spectral energy is therefore $2\rho c^2\epsilon^2q^{+-}(\mathbf{k})$. For convenience we denote $q^{+-}(\mathbf{k})$ as $e(\mathbf{k})$.

To leading order in ϵ, $q^{ss'}(\mathbf{k},\mathbf{k}')$ and $q^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')$ [which we may call $q_0^{ss'}(\mathbf{k},\mathbf{k}')$ and $q_0^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')$] are independent of time. Anticipating, however, that certain parts of the higher order iterates in their asymptotic expansions may become unbounded, we will allow both $q_0^{ss'}(\mathbf{k},\mathbf{k}')$ and $q_0^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')$ to be slowly varying in time,

$$\begin{aligned}\frac{dq_0^{ss'}(\mathbf{k},\mathbf{k}')}{dt}&=\epsilon^2F_2^{ss'},\\\frac{dq_0^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')}{dt}&=\epsilon^2F_3^{ss's''}\end{aligned} \tag{3.8}$$

and we will choose F_2 and F_3 to remove those terms with unbounded growth from the later iteration. We will find that for $s'=-s$, F_2^{s-s} is given by the right-hand side of the acoustic KE,

$$\begin{aligned}F_2^{ss'}=q_0^{s\,s'}(\mathbf{k},\mathbf{k}')&\lim_{\epsilon^2\to0}\sum_{s_ps_q}\int\frac{s_qs_p}{s\omega}\int(L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q})^2q^{s_p,-s_p}(\mathbf{k}_p)\\&\times\Delta(s_p\omega_p+s_q\omega_q-s\omega)\delta(\mathbf{k}_p+\mathbf{k}_q-\mathbf{k})d\mathbf{k}_pd\mathbf{k}_q\\&+\lim_{\epsilon^2\to0}\sum_{s_ps_q}\int\frac{s_qs_p}{s\omega}\int(L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q})^2q_0^{s\,s'}(\mathbf{k}_q,-\mathbf{k}_q)q^{s_p,-s_p}\\&\times(\mathbf{k}_p)\Delta(s_p\omega_p+s_q\omega_q-s\omega)\delta(\mathbf{k}_p+\mathbf{k}_q-\mathbf{k})d\mathbf{k}_pd\mathbf{k}_q,\end{aligned}$$

and that F_2^{ss} and $F_3^{ss's''}$ have the forms

$$iq_0^{ss}(\mathbf{k},\mathbf{k}')(\bar\Omega^s_{\mathbf{k}}+\bar\Omega^s_{\mathbf{k}'}) \tag{3.9}$$

and

$$iq_0^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')(\bar\Omega^s_{\mathbf{k}}+\bar\Omega^{s'}_{\mathbf{k}'}+\bar\Omega^{s''}_{\mathbf{k}''}), \tag{3.10}$$

respectively. It is clear that $\bar\Omega^s_k$ can be interpreted as a complex frequency modification. Its exact expression is given by

$$\bar\Omega^s_{\mathbf{k}}=-4i\lim_{\epsilon^2\to0}\sum_{s_ps_q}\int\frac{s_qs_p}{s\omega}\int(L^{s,s_p,s_q}_{\mathbf{k},\mathbf{k}_p,\mathbf{k}_q})^2q^{s_p,-s_p}(\mathbf{k}_p)$$

$$\Delta(s_p\omega_p+s_q\omega_q-s\omega)\delta(\mathbf{k}_p+\mathbf{k}_q-\mathbf{k})d\mathbf{k}_p d\mathbf{k}_q, \tag{3.11}$$

and, when calculated, is precisely equal to $s(\omega-c|\mathbf{k}|)\epsilon^2$ in Eq. (1.17). Note that, in Eq. (3.11), $t=T\epsilon^2$ and T is finite. The $\ln(1/\epsilon^2)$ coefficient comes from the term $\ln t$ or $\ln(T/\epsilon^2)=\ln T+\ln(1/\epsilon^2)$ in the asymptotic expansion. For finite T, the dominant part is $\ln(1/\epsilon^2)$.

The perturbations method has the advantage that it is relatively simple to execute. However, there is no *a priori* guarantee that terms appearing later in the formal series cannot have time dependencies, which mean they affect the leading approximations on time scales comparable to or less than ϵ^{-2} (e.g., a term $\epsilon^4 t^3$ should be accounted for before the term $\epsilon^2 t$). To check this, one must have a systematic approach for exploring all orders in the formal perturbation series, and removing (renormalizing) in groups those resonances which make their cumulative effects at time scales $\epsilon^{-N}(\ln(1/\epsilon)^{-M})$, $N,M=1,2,3,\ldots$. The diagram approach, which requires some familiarity to execute, is designed to do this and, both for completeness and the fact that we will have to proceed beyond the one-loop approximation to resolve the questions of the angular redistribution of spectral energy, we include it here.

IV. DIAGRAMMATIC APPROACH TO ACOUSTIC TURBULENCE

A. Objects of diagrammatic technique

Let us define the "bare" Green's function of Eq. (2.24) as

$$G_0(\mathbf{k})=\frac{1}{\omega-ck+i0}. \tag{4.1}$$

One may see from Eq. (2.24) that this function describes the response of the system of noninteracting acoustic waves on some external force. In the denominator we added the term $+i0$ by requirement of causality. We remark that causality (the arrow of time) is introduced in the perturbation approach by the limit $t\to\infty$, and the fact that $\mathrm{sgn}t$ appears in Eq. (1.7). Next we introduce the "dressed" Green function, which is the response of interacting wave systems on this force:

$$(2\pi)^4 G(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')=\left\langle\frac{\delta b(\mathbf{k},\omega)}{\delta f(\mathbf{k}',\omega')}\right\rangle. \tag{4.2}$$

We will also be interested in the double correlation function $n(\mathbf{k},\omega)$ of the acoustic field b,b^*,

$$(2\pi)^4 n(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')=\langle b(\mathbf{k},\omega)b^*(\mathbf{k}',\omega')\rangle. \tag{4.3}$$

The simultaneous double correlator of the acoustic field $n(\mathbf{k})$ is determined by

$$(2\pi)^3 n(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')=\langle b(\mathbf{k},t)b^*(\mathbf{k}',t)\rangle. \tag{4.4}$$

This is related to the different-time correlators in the ω representation $n(\mathbf{k},\omega)$ as follows:

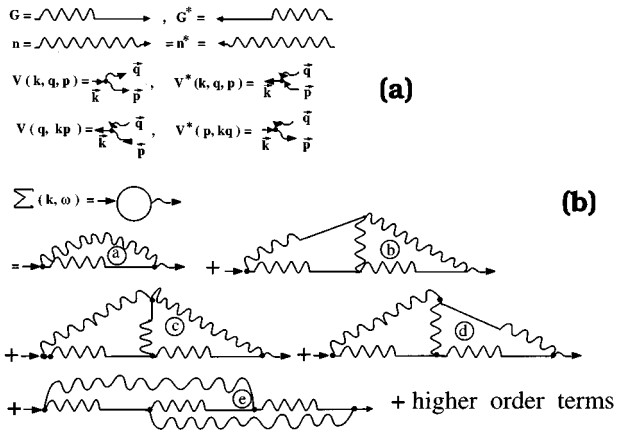

FIG. 1. (a) Basic objects of diagrammatic pertubation approach. (b) First terms in the expansion of mass operator $\Sigma(\mathbf{k},\omega)$.

$$n(\mathbf{k})=\int n(\mathbf{k},\omega)\frac{d\omega}{2\pi}. \tag{4.5}$$

The Green's and correlation functions together with the bare vertex $V(\mathbf{k},\mathbf{q},\mathbf{p})$, Eq. (2.20), are the basic objects of diagrammatic perturbation approach which we are going to use [see Fig. 1(a)].

B. Dyson-Wyld equations

In the diagrammatic series for the Green's function, one may perform the partial Dyson's summation over one-particle irreducible diagrams. This results in the Dyson equation for the Green's functions,

$$G(\mathbf{k},\omega)=\frac{1}{\omega-\omega_0(k)+i0-\Sigma(\mathbf{k},\omega)}, \tag{4.6}$$

where the "mass operator" $\Sigma(\mathbf{k},\omega)$ gives the nonlinear correction to the complex frequency $\omega_0(\mathbf{k})+i0$ due to the interaction (2.19). This is an infinite series with respect to the bare amplitude $V(\mathbf{k},\mathbf{q},\mathbf{p})$, Eq. (2.20), dressed Green's function (4.2) and double correlation function $n(\mathbf{k},\omega)$, Eq. (4.3). All of the contributions of the second and fourth orders in V are shown in Fig. 1(b).

We have not specified the direction of arrows in Fig. 1(b); each diagram should be interpreted as a sum of diagrams with all possible directions of arrows compatible with vortex $V(\mathbf{k},\mathbf{q},\mathbf{p})$, describing the three-wave processes $1\leftrightarrow 2$. For example, diagram (a) in Fig. 1(b) corresponds to three diagrams shown in Fig. 2. The diagram $(a4)$ on Fig. 2 describes the nonresonant process $0\leftrightarrow 3$, which is not essential for our consideration.

With the help of the similar Dyson's summing of one-particle irreducible diagrams, one can derive Wyld's equation for $n(\mathbf{k},\omega)$:

$$n(\mathbf{k},\omega)=|G(\mathbf{k},\omega)|^2[D(\mathbf{k},\omega)+\Phi(\mathbf{k},\omega)]. \tag{4.7}$$

Here $D(\mathbf{k},\omega)$ is the correlation function of the white noise,

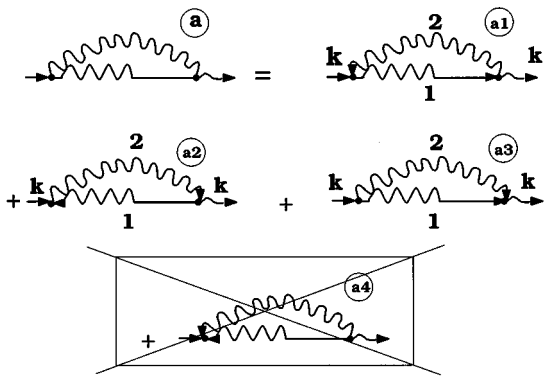

FIG. 2. Diagrams (a) from Fig. 1 with specified directions of arrows.

$$(2\pi)^4 D(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')=\langle f(k\omega)f^*(k'\omega')\rangle, \tag{4.8}$$

and the mass operator $\Phi(\mathbf{k},\omega)$ describes the nonlinear corrections to $D(\mathbf{k},\omega)$. This is an infinite series with respect to the same objects $G(\mathbf{k},\omega)$,$n(\mathbf{k},\omega)$, and $V(\mathbf{k},\mathbf{q},\mathbf{p})$. All diagrams of the second and fourth orders are shown in Fig. 3(a).

We also have not specified arrow directions in the diagrams for $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$. In complete analogy with diagrams for $G(\mathbf{k},\omega)$, one diagram in Fig. 3(a) corresponds to *two* diagrams ($a1$) and ($a2$) in Fig. 3(b). All the rest diagrams for $\Phi(\mathbf{k},\omega)$ reproduce in the same way—one chooses all possible directions of arrows, and discards those which are incompatible with the definition of vertex V [see Fig. 1(a)].

C. One-pole approximation

1. Green's function

We have assumed from the beginning that the wave amplitude is small. Therefore,

$$\Sigma(\mathbf{k},\omega)\ll\omega_0(k). \tag{4.9}$$

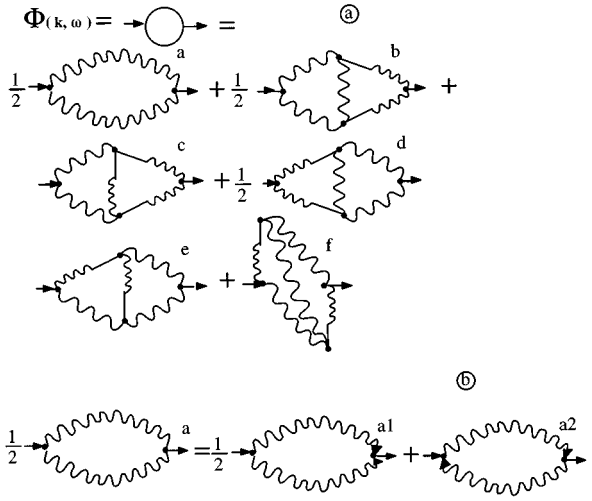

FIG. 3. First terms in the diagrammatic pertubation expansion for mass operator $\Psi(\mathbf{k},\omega)$.

As a result, the Green's function has a sharp peak in the vicinity of $\omega=ck$, and one may (as a first step in the analysis) neglect the ω dependence of $\Sigma(\mathbf{k},\omega)$ and put

$$\Sigma(\mathbf{k},\omega)\simeq\Sigma(\mathbf{k},\omega\simeq c\mathbf{k}). \tag{4.10}$$

The validity of this assumption will be checked later. Under this assumption the Green's function (4.2) has a simple one-pole structure,

$$\tilde{G}(\mathbf{k},\omega)=\frac{1}{\omega-\omega(\mathbf{k})+i\gamma(\mathbf{k})}, \tag{4.11}$$

where

$$\omega(\mathbf{k})=\omega_0(\mathbf{k})+\mathrm{Re}\Sigma(\mathbf{k},\omega_*), \tag{4.12}$$

$$\gamma(\mathbf{k})=-\mathrm{Im}\Sigma(\mathbf{k},\omega_*). \tag{4.13}$$

Now we have to decide how to choose ω_* "in the best way." The simplest way is to put $\omega_*=\omega_0(\mathbf{k})=ck$, as was stated in Eq. (4.10). As a next step we can take a "more accurate" expression $\omega_*=\omega(\mathbf{k})$, i.e., to take into account the real part of correction to $\omega_0(\mathbf{k})$. But later we will see that a better choice is

$$\omega_*=\omega(\mathbf{k})+i\gamma(\mathbf{k}), \tag{4.14}$$

which is consistent with the position of the pole of $\tilde{G}_*(\mathbf{k},\omega)$. We will show that this choice is self-consistent, while deriving the balance equation in Sec. V C.

2. Double correlation function

The same type of approximation may be performed for the correlation function. That is, in the Wyld equation (4.7) one may replace $G(\mathbf{k},\omega)$ by $\tilde{G}(\mathbf{k},\omega)$, and neglect the ω dependence of $\Phi(\mathbf{k},\omega)$ by putting $\Phi(\mathbf{k},\omega)\to\tilde{\Phi}(k)=\Phi(\mathbf{k},\omega_*)$, or

$$\tilde{n}(\mathbf{k},\omega)=|\tilde{G}(\mathbf{k},\omega)|^2[D(k)+\tilde{\Phi}(k)], \tag{4.15}$$

We will call this *one-pole approximation for the correlation function.*

D. One-loop approximation

Let us begin our treatment with the simple one-loop (or direct interaction) approximation for mass operators Σ and Φ. This approximation corresponds to taking into account just the second order [in bare vertex V, Eq. (2.20)] diagrams for the mass operators Σ and Φ. The two-loop approximation will be considered in Appendix C. We will estimate two-loop diagrams, and will show that some of them give the same order contribution to γ_k as one-loop diagrams. Therefore, the one-loop approximation is an uncontrolled approximation, but we believe that it gives qualitatively correct results. Note that these diagrams include the dressed Green's function, in contrast to the approximation of the kinetic equation which is nothing but a one-loop approximation with the bare Green's function inside. We will see below that this difference is very important in the particular case of acoustic turbulence. The KE for waves with a linear dispersion law forbids the angular evolution of energy because conservation

laws of energy and momentum allow interaction only for waves with parallel wave vectors. In the one-loop approximation with a dressed Green's function, the conservation laws $\omega(\mathbf{k})\pm\omega(\mathbf{k}_1)=\omega(\mathbf{k}\pm\mathbf{k}_1)$ are satisfied with some accuracy [of the order of $\gamma(\mathbf{k})$]. As a result, there exists a cone of allowed angles between $\mathbf{k}$ and $\mathbf{k}_1$ in which interactions are allowed. Therefore one has to expect some angle evolution of wave packages within this approximation. Combining Eq. (4.11) with Eq. (4.15), one has the following expression:

$$n(\mathbf{k},\omega)=\frac{2\gamma(\mathbf{k})\tilde{n}(\mathbf{k})}{[\omega-\omega(\mathbf{k})]^2+\gamma^2(k)}. \tag{4.16}$$

1. Calculations of $\Sigma(\mathbf{k},\omega)$

In the one-loop approximation, the expression for $\Sigma(\mathbf{k},\omega)$ has the form

$$\Sigma(\mathbf{k},\omega)=\Sigma_{a1}(\mathbf{k},\omega)+\Sigma_{a2}(\mathbf{k},\omega)+\Sigma_{a3}(\mathbf{k},\omega), \tag{4.17}$$

where $\Sigma_j(\mathbf{k},\omega)$ is given by Eqs. (A1)–(A3). Our goal here is to analyze these expressions in the one-pole approximation, by substituting in it "one-pole' expressions $n(\mathbf{k},\omega)$ and $G(\mathbf{k},\omega)$ from Eqs. (4.11) and (4.16). In the resulting expression one can perform the integration over ω analytically. The result is

$$\Sigma(\mathbf{k},\omega)=\int\frac{d^3k_1d^3k_2}{(2\pi)^3}\left(\frac{|V(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1)|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)[n(\mathbf{k}_1)-n(\mathbf{k}_2)]}{\omega+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)+i(\gamma_1+\gamma_2)}+\frac{|V(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2)|^2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)n(\mathbf{k}_2)}{\omega-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)+i(\gamma_1+\gamma_2)}\right). \tag{4.18}$$

Next we introduce $\Sigma(\mathbf{k})=\Sigma(\mathbf{k},\omega_*)$, with ω_* given by Eq. (4.14), and consider Eq. (4.18) in the limit of small γ, which allows us to perform analytically integrations over perpendicular components of wave vectors. The result for the damping frequency $\gamma(k)$ may be represented in the following form (for details, see Appendix B):

$$\gamma(k)=\frac{A^2k^2}{4\pi c}\int_{1/L}^{\infty}n(q)q^2dq\simeq\frac{A^2k^2}{4\pi c}N(\Omega). \tag{4.19}$$

Here we introduced a cutoff for small k at $1/L$, where L is the size of the box. We also introduced "the density of the number of particles" $N(\Omega)$ in the solid angle according to

$$N(\Omega)=\int k^2n(\mathbf{k})dk, \tag{4.20}$$

such that the total number of particles

$$N=\int N(\Omega)d\Omega. \tag{4.21}$$

After substituting A from Eq. (B11), one has the following estimate for $\gamma(\mathbf{k})$:

$$\gamma(\mathbf{k})\simeq k^2N(\Omega)/\rho_0, \tag{4.22}$$

Consider now $\Sigma'(\mathbf{k})\equiv\mathrm{Re}\Sigma(\mathbf{k})$. It follows from Eq. (B12) that

$$\begin{aligned}\Sigma(\mathbf{k})&=\frac{A^2}{4\pi^2c}\int dq\int_0^{y_{\max}}dy\;q^2n(q)\frac{y}{y^2+\Gamma_{k12}^2}\\&\quad\times[(k^2+2kq+q^2)-(k^2-2kq+q^2)]\\&\simeq\frac{A^2k}{\pi^2c^2}\int dq\int_0^{y_{\max}}\frac{ydy}{y^2+\Gamma_{k12}^2}[cq^3n(q)],\end{aligned} \tag{4.23}$$

where $\Gamma_{k12}=\gamma(k)+\gamma(k_1)+\gamma(k_2)$ is the "triad interaction" frequency. One may evaluate the integral with respect to y as

$$Ł(q)=\ln\frac{y_{\max}}{\Gamma_{kkq}}\simeq\ln\frac{ck^2}{q\gamma(\mathbf{k})}. \tag{4.24}$$

After substituting $\gamma(\mathbf{k})$ from Eq. (4.22), one has

$$Ł(q)\propto\ln\rho_0/qN(\Omega). \tag{4.25}$$

The main contribution to integral (4.23) over q comes from the infrared region $q\simeq1/L$. This gives the estimate

$$\Sigma'(\mathbf{k})=\frac{A^2k}{\pi^2c^2}ŁE(\Omega), \tag{4.26}$$

where we have defined the density of the wave energy in solid angle as

$$E(\Omega)=\int\omega_0(k)n(k)k^2dk. \tag{4.27}$$

This value relates to $N(\Omega)$ as follows:

$$E(\Omega)\simeq\frac{c}{L}N(\Omega). \tag{4.28}$$

Equation (4.26) together with the expression (B11) for A may be written as

$$\Sigma'(\mathbf{k})\simeq ck\epsilon\,\ln1/\epsilon, \tag{4.29}$$

where

$$\epsilon\simeq E(\Omega)/\rho_0c^2 \tag{4.30}$$

is the dimensionless parameter of nonlinearity, the ratio of energy of acoustic turbulence, and the density of thermal energy of media $\rho_0c^2\simeq\tilde{n}T$, where $\tilde{n}$ is the concentration of atoms.

Equation (4.22) for $\gamma(\mathbf{k})$ may be written in a similar form,

$$\gamma(\mathbf{k})\simeq ck(kL)\epsilon. \tag{4.31}$$

One can see that

$$\frac{\gamma(\mathbf{k})}{\Sigma'(\mathbf{k})} \propto \frac{kL}{\ln 1/\epsilon}. \tag{4.32}$$

This means that, for a large enough inertial interval,

$$\gamma(\mathbf{k}) \gg \Sigma'(\mathbf{k}), \tag{4.33}$$

and one may neglect the nonlinear corrections $\Sigma'(\mathbf{k})$ to the frequency with respect to the damping of the waves $\gamma(\mathbf{k})$. This shows that our above calculations of $\Sigma(\mathbf{k})$ is self-consistent. Later we also will take into account only damping $\gamma(\mathbf{k})$ in the expressions for the Green's functions, taking $\omega(k)=\omega_0(k)=ck$.

2. Calculations of $\Phi(\mathbf{k},\omega)$

In the one-loop approximation expression for $\Phi(\mathbf{k},\omega)$ has the form (A4). After substitution of $n(\mathbf{k},\omega)$ in the one-pole approximation (4.16) one may perform analytically integration over frequencies:

$$\Phi(\mathbf{k},\omega)=\int \frac{d^3k_1 d^3k_2}{(2\pi)^3} n(\mathbf{k}_1)n(\mathbf{k}_2) \times\left[\frac{|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2(\gamma(\mathbf{k}_1)+\gamma(\mathbf{k}_2))\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{[\omega-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]^2+[\gamma(\mathbf{k}_1)+\gamma(\mathbf{k}_2)]^2} + \frac{|V(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})|^2(\gamma(\mathbf{k}_1)+\gamma(\mathbf{k}_2))\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)}{[\omega+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]^2+[\gamma(\mathbf{k}_1)+\gamma(\mathbf{k}_2)]^2}\right]. \tag{4.34}$$

We will analyze this expression in Sec. IV D 3.

3. Balance equation

Consider the Dyson-Wyld equations (4.6) and (4.7) in the inertial interval, where one can neglect $\gamma_0(\mathbf{k})$ in comparison with $\mathrm{Im}\Sigma(\mathbf{k},\omega)$ and $D(\mathbf{k})$ in comparison with $\Phi(\mathbf{k},\omega)$:

$$G(\mathbf{k},\omega)=\frac{1}{\omega-\omega_0(\mathbf{k})-\Sigma(\mathbf{k},\omega)}, \tag{4.35}$$

$$n(\mathbf{k},\omega)=|G(\mathbf{k},\omega)|^2\Phi(\mathbf{k},\omega). \tag{4.36}$$

It follows from Eq. (4.6) that

$$\mathrm{Im}G(\mathbf{k},\omega)=|G(\mathbf{k},\omega)|^2\mathrm{Im}\Sigma(\mathbf{k},\omega). \tag{4.37}$$

By comparing Eqs. (4.36) and (4.37), one may see that the combination

$$L(\mathbf{k},\omega)\equiv\Phi(\mathbf{k},\omega)\mathrm{Im}G(\mathbf{k},\omega)-n(\mathbf{k},\omega)\mathrm{Im}\Sigma(\mathbf{k},\omega) \tag{4.38}$$

is equal to zero. In particular,

$$L(\mathbf{k})\equiv\int L(\mathbf{k},\omega)\frac{d\omega}{2\pi}=0. \tag{4.39}$$

Together with Eq. (4.38), this gives

$$\mathrm{Im}\int\frac{d\omega}{2\pi}[\Phi(\mathbf{k},\omega)G(\mathbf{k},\omega)-n(\mathbf{k},\omega)\Sigma(\mathbf{k},\omega)]=0. \tag{4.40}$$

Let us now compute the first term in Eq. (4.40). By substituting Eq. (4.34) for $\Phi(\mathbf{k},\omega)$, and Eq. (4.11) and integration over ω, one has

$$\int\frac{d\omega}{2\pi}G(\mathbf{k},\omega)\Phi(\mathbf{k},\omega)=\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^3}n(\mathbf{k}_1)n(\mathbf{k}_2)\left[\frac{1}{2}\frac{|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{\omega_0(\mathbf{k})-\omega_0(\mathbf{k}_1)-\omega_0(\mathbf{k}_2)-i\Gamma_{k12}}+\frac{|V(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)}{\omega_0(\mathbf{k})+\omega_0(\mathbf{k}_1)-\omega_0(\mathbf{k}_2)-i\Gamma_{k12}}\right]. \tag{4.41}$$

Next we will perform integration over ω in Eq. (4.40). Remember that $\Sigma(\mathbf{k},\omega)$ is an analytical function in the upper half plane of ω, while $n(\mathbf{k},\omega)$ has one pole there. Therefore,

$$\mathrm{Im}\int\frac{d\omega}{2\pi}n(\mathbf{k},\omega)\Sigma(\mathbf{k},\omega)=n(\mathbf{k})\mathrm{Im}\Sigma(\mathbf{k},\omega_*) \tag{4.42}$$

where ω_* is given by Eq. (4.14). This is the justification of our choice ω_*.

Now let us put everything together to obtain

$$0=L(\mathbf{k})=\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^3}\Gamma_{k12}\left\{\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{1}{2}\frac{|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2\{n(\mathbf{k}_1)n(\mathbf{k}_2)-n(\mathbf{k})[n(\mathbf{k}_1)+n(\mathbf{k}_2)]\}}{(\omega_0(\mathbf{k})-\omega_0(\mathbf{k}_1)-\omega_0(\mathbf{k}_2))^2+\Gamma_{k12}^2}+\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) \times\frac{|V(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})|^2\{n(\mathbf{k}_2)[n(\mathbf{k}_1)+n(\mathbf{k})]-n(\mathbf{k})n(\mathbf{k}_1)\}}{(\omega_0(\mathbf{k})+\omega_0(\mathbf{k}_1)-\omega_0(\mathbf{k}_2))^2+\Gamma_{k12}^2}\right\}. \tag{4.43}$$

This is the main result of the diagrammatic approach: the *balance equation for stationary in time acoustic turbulence*. In the nonstationary case one can similarly obtain the *generalized kinetic equation* in the form

$$\frac{\partial n(\mathbf{k},t)}{\partial t}=L(\mathbf{k},t), \tag{4.44}$$

where $L(\mathbf{k},t)$ is given by Eq. (4.43) with a correlator depending on time $n(\mathbf{k}_j)\to n(\mathbf{k}_j,t)$.. In the limit $\gamma(\mathbf{k})\to 0$, this expression turns into the well known (cf. [1]) collision integral for the three-wave kinetic equation

$$\begin{aligned}\mathcal{S}t\{n(\mathbf{k},t)\}=\lim_{\Gamma_{k12}\to 0}L(\mathbf{k})=2\pi\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^3}\frac{1}{2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ \times|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2\{n(\mathbf{k}_1)n(\mathbf{k}_2)-n(\mathbf{k})\\ \times[n(\mathbf{k}_1)+n(\mathbf{k}_2)]\}\delta[\omega_0(\mathbf{k})-\omega_0(\mathbf{k}_1)\\ -\omega_0(\mathbf{k}_2)]\\ +\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)|V(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})|^2\\ \times\{n(\mathbf{k}_2)[n(\mathbf{k}_1)+n(\mathbf{k})]-n(\mathbf{k})n(\mathbf{k}_1)\}\\ \times\delta[\omega_0(\mathbf{k})+\omega_0(\mathbf{k}_1)-\omega_0(\mathbf{k}_2)].\end{aligned} \tag{4.45}$$

We see that the generalized kinetic equation differs from the well known collision term in the three-wave kinetic equation by replacing δ functions on the corresponding Lorenz function with the width of the Γ_{k12}-triad interaction frequency.

V. CONCLUSION

In the present paper we have begun to develop a consistent statistical description of acoustic turbulence based both on the long time asymptotic analyses (Sec. III) and on the perturbation diagrammatic approach (Sec. IV). The first approach is more straightforward. The diagrammatic approach provides a systematic way of analyzing higher order terms in the perturbation theory.

Our main result is that nonlinear corrections to the frequency are much smaller than the nonlinear damping of the waves. We also find the balance equation (4.43), which generalizes the simple kinetic equation for acoustic waves. One can show that the balance equation (4.43) has the same isotropic solution (Zakharov-Sagdeev spectrum) as the kinetic equation. However, the kinetic equation for acoustic turbulence does not describe the angle evolution of turbulence: any arbitrary angle distribution is the solution of KE. In contrast, our balance equation (4.43) contains terms which describe an angular redistribution of the energy because of the nonzero value of the interaction cone, which is proportional to Γ_{k12}. However, we have yet to show that this expression contains all such terms to this order.

One may imagine three very different ways of the angle evolution of anisotropic acoustic turbulence. The first one is a tendency to form very narrow beams with a characteristic width of about one interaction angle. The second one is an approach to isotropy downstream from the large wave vectors. The last possibility is to form a beam with a characteristic width of about unity, exactly as it happens in the turbulence of waves with weak dispersion [17]. Another important question is, do the spectra of acoustic turbulence depend on the features of pumping or they are universal (independent of details of energy influx)? We intend to answer these questions (in the framework approximations we made in that paper) in our next project. It is an exciting challenge to try to go beyond the approximations made here in order to understand whether the scaling index of the interaction vertex in the system of acoustic waves in two- and three-dimensional media must be renormalized or not.

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APPENDIX A: RULES FOR WRITING AND READING OF DIAGRAMS FOR MASS OPERATORS

Here we state without proof the set of rules for writing down diagrammatic series.

(1) In order to write down all diagrams for Σ and Φ of $2n$ order in vertices, one should draw $2n$ vertices and connect them with each other by lines n and G in all possible ways. Two ends must be left free. If both ends are straight, we will obtain a diagram for $\Phi(\mathbf{k},\omega)$; if one of them is wavy, this will be a diagram for $\Sigma(\mathbf{k},\omega)$.

(2) The diagrams for Φ and Σ containing closed loops in Green's function (GF) are absent. This follows from the fact that the Wyld's diagrammatic technique (DT) appears from glued trees.

(3) There is no mass operator with two wavy ends in DT.

(4) In the diagrams for Φ (for Σ) one can pass from every vertex along the G lines to the entrance and exit in a single way.

(5) In every diagram for Σ there is a single root linking the entrance and exit along the G lines – the backbone of the diagram. The rest G lines of the diagrams may be called the rips.

(6) The diagrams for Φ contain a basic cross section in which they may be cut in a single way into two parts only at lines $n(\mathbf{k},\omega)$.

(7) Every V vertex is entered by one arrow and exited by two. The V^* vertex is entered by two arrows and exited by one.

One can show (see [12]) that rules (3)–(7) follows from (1) and (2). The rules of reading diagrams are the follows:

(1) Write down the product of DT objects (double correlator, Green function, or vertex) (with corresponding arguments) corresponding to each element of the diagram.

(2) Write down δ functions in the 4-momenta for $2n-1$ vertices in such a way that the sum of entering the 4-momenta is equal to the sum of exiting them. One of the vertices (for, example the one corresponding to the end of the diagram) does not contain the δ function.

(3) Perform an integration along all internal lines of dia-

gram : $di=[dk_i/(2\pi)^d]d\omega_i/(2\pi)$, where d is the space dimension.

(4) Multiply the diagram by $(2\pi)^{(d+1)}$.

(5) Multiply the diagram by $1/p$, where p is the number of elements in its symmetry group. For example diagrams $(a1)$, $(a2)$, and $(a3)$ correspond to the following analytical expressions q:

$$\Sigma_{a1}(\mathbf{k},\omega)=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\frac{d\omega_1d\omega_2}{2\pi}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\times\delta(\omega+\omega_1-\omega_2)|V(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1)|^2G_2n_1, \tag{A1}$$

$$\Sigma_{a2}(\mathbf{k},\omega)=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\frac{d\omega_1d\omega_2}{2\pi}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\times\delta(\omega+\omega_1-\omega_2)|V(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1)|^2G_1^*n_2, \tag{A2}$$

$$\Sigma_{a3}(\mathbf{k},\omega)=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\frac{d\omega_1d\omega_2}{2\pi}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\times\delta(\omega-\omega_1-\omega_2)|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2G_1n_2. \tag{A3}$$

Here we defined the following shorthand notation: $G_j=G(\mathbf{k}_j,\omega_j)$, and $n_i=n_i(\mathbf{k}_i,\omega_i)$. In the same way one can find analytical expressions for $\Phi_a(\mathbf{k},\omega)$:

$$\Phi(\mathbf{k},\omega)=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\frac{d\omega_1d\omega_2}{2\pi}\Big[\frac{1}{2}|V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)|^2n_1n_2\times\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega-\omega_1-\omega_2)+|V(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})|^2n_1n_2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\times\delta(\omega+\omega_1-\omega_2)\Big]. \tag{A4}$$

Analytical expressions for s in fourth order diagrams (two-loop diagrams) will be shown in Appendix C.

APPENDIX B: CALCULATION OF $\Sigma(\mathbf{k},\omega)$-DETAILS

Let us start from Eq. (4.18) and introduce $\Sigma(\mathbf{k})=\Sigma(\mathbf{k},\omega_*)$ with ω_* given by Eq. (4.14):

$$\Sigma(\mathbf{k})=\int \frac{d^3k_1d^3k_2}{(2\pi)^3}\Big(\frac{|V(\mathbf{k}_2,\mathbf{k},\mathbf{k}_1)|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)[n(\mathbf{k}_1)-n(\mathbf{k}_2)]}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)+i\Gamma_{k12}}+\frac{|V(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2)|^2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)n(\mathbf{k}_2)}{\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)+i\Gamma_{k12}}\Big), \tag{B1}$$

where

$$\Gamma_{k12}=\gamma(\mathbf{k})+\gamma(\mathbf{k}_1)+\gamma(\mathbf{k}_2) \tag{B2}$$

is the "triad-interaction" frequency and $1/\Gamma_{k12}$ is the triad interaction time. One can consider Eqs. (B1) and (B2) as integral equations for the damping of wave $\gamma(\mathbf{k})=-\mathrm{Im}\Sigma(\mathbf{k})$ and for the frequency $\omega(\mathbf{k})=\omega_0(\mathbf{k})+\mathrm{Re}\Sigma(\mathbf{k})$.

First we consider these equations in the limit of weak interaction where $\Gamma\to0$, and the main contribution to the first term in Eq. (B1) comes from the region where

$$\omega(\mathbf{k})+\omega(\mathbf{k}_1)=\omega(\mathbf{k}_2),\quad \mathbf{k}+\mathbf{k}_1=\mathbf{k}_2. \tag{B3}$$

These are conservation laws for three-wave confluence processes $0+1\to2$. The main contribution for the second term in Eq. (B1) comes from the region

$$\omega(\mathbf{k})=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2),\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2. \tag{B4}$$

These are conservation laws for decays processes $0\to1+2$. For weak interaction one may replace $\omega(\mathbf{k})$ on $\omega_0(\mathbf{k})=ck$ in Eqs. (B3) and (B4). Than it follows from Eqs. (B3) and (B4) that $\mathbf{k}_1\|\mathbf{k}_2\|\mathbf{k}$, with $\mathbf{k}_1,\mathbf{k}_2$ directed along $\mathbf{k}$. This fact makes it natural to introduce new variables in integrals (B1): the scale positive variable $q>0$, and two-dimensional vector $\boldsymbol{\kappa}$, such that

$$\mathbf{k}_1=q\mathbf{k}/k+\boldsymbol{\kappa},\quad \boldsymbol{\kappa}\perp\mathbf{k}. \tag{B5}$$

In the first term of Eq. (B1),

$$\mathbf{k}_2=(k+q)\mathbf{k}/k+\boldsymbol{\kappa},\quad 0\leqslant q. \tag{B6}$$

In the second term,

$$\mathbf{k}_2=(k-q)\mathbf{k}/k-\boldsymbol{\kappa},\quad 0\leqslant q\leqslant k. \tag{B7}$$

For $\kappa\ll k$ the denominators in integrals (B1) strongly depend on κ. Indeed,

$$\omega_0(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(|\mathbf{k}+\mathbf{k}_1|)=ck\frac{\kappa^2}{2q(k+q)}, \tag{B8}$$

$$\omega_0(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(|\mathbf{k}-\mathbf{k}_1|)=-ck\frac{\kappa^2}{2q(k-q)}. \tag{B9}$$

This allows us to neglect the κ dependence of interaction $V(\mathbf{k},\mathbf{q},\mathbf{p})$ and correlation $n(\mathbf{k}_i)$ in the numerator of (B1) for estimation. The result is

$$\Sigma(\mathbf{k})=\frac{A^2k}{8\pi^2}\int_0^{k^2}d\kappa^2\Big[\int_0^\infty dq\frac{q(k+q)[n(q)-n(k+q)]}{ck\kappa^2/[2q(k+q)]+i\Gamma_{k12}}+\int_0^k dq\frac{q(k-q)n(q)}{-ck\kappa^2/[2q(k-q)]+i\Gamma_{k12}}\Big], \tag{B10}$$

where

$$A=3(g+1)\sqrt{c/4\pi^3\rho_0} \tag{B11}$$

is a factor in Eq. (2.20) so that, for parallel or almost parallel wave vectors, $V(\mathbf{k},\mathbf{q},\mathbf{p})=A\sqrt{kqp}$. After changing variables, this integral becomes more transparent,

$$\Sigma(\mathbf{k})=\frac{A^2}{4\pi^2 c}\left[\int dq\int_0^{y_{\max}}dy q^2(k+q)^2\frac{[n(q)-n(k+q)]}{y+i\Gamma_{k12}}\right.$$
$$\left.-\int_0^{y_{\max}}dy\int_0^k dq\frac{q^2(k-q)^2 n(q)}{y-i\Gamma_{k12}}\right]. \tag{B12}$$

One may estimate $y_{\max}\simeq ck^2/2q$ from the fact that our expressions were obtained by expanding in κ/k; therefore they should be at least $\kappa<k$.

Now let us consider the imaginary and real parts of Σ separately. It is convenient to begin with $\gamma(\mathbf{k})=-\operatorname{Im}\Sigma(\mathbf{k})$:

$$\gamma(\mathbf{k})\simeq\frac{A^2}{4\pi^2 c}\int_0^\infty dy\left[\int_0^\infty dq\ q^2(k+q)^2\right.$$
$$\times\Gamma_{k12}\frac{[n(q)-n(k+q)]}{y^2+\Gamma_{k12}^2}$$
$$\left.+\int_0^k dq\frac{q^2(k-q)^2 n(q)\Gamma_{k12}}{y^2+\Gamma_{k12}^2}\right]. \tag{B13}$$

Here we changed the upper limit of integration, $y_{\max}\to\infty$, because the main contribution to the integral comes from the area $y\simeq\Gamma\ll ck$. After trivial integration with respect to y, one has

$$\gamma(\mathbf{k})\simeq\frac{A^2}{8\pi c}\left[\int_0^\infty q^2(k+q)^2[n(q)-n(k+q)]dq\right.$$
$$\left.+\int_0^k q^2(k-q)^2 n(q)dq\right]. \tag{B14}$$

This expression for $\gamma(\mathbf{k})$ corresponds to that given by the kinetic equation [1] for waves. For further progress it is necessary to make some assumption about $n(q)$. Let us assume that $n(q)$ vanishes with growing of q faster than $1/q^4$. [Remember, that in the Zakharov-Sagdeev spectrum $n(q)\propto q^{-9/2}$, and in the Kadomtsev-Petviashvili spectrum $n(q)\propto q^{-4}$. This assumption is true for the Zakharov-Sagdeev spectrum, and is not true for the Kadomtsev-Petviashvili one.] For such spectra the main contribution to the integral comes from small $q\ll k$. In this case contributions from first and second integrals in Eq. (B14) coincides, and may be represented in the form

$$\gamma(k)=\frac{A^2k^2}{4\pi c}\int_{1/L}^\infty n(q)q^2dq\simeq\frac{A^2k^2}{4\pi c}N(\Omega). \tag{B15}$$

APPENDIX C: ESTIMATION OF THE TWO-LOOP DIAGRAMS

Let us write down analytical expression which correspond to one of the diagrams (b) in Fig. 1(b),

$$\Sigma_b(\mathbf{k},\omega)=\int\frac{d\mathbf{k}_1 d\mathbf{k}_2 d\omega_1 d\omega_2}{(2\pi)^8}V_aV_bV_cV_d n(k_1,\omega_1)$$
$$\times n(k_2,\omega_2)G(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2)$$
$$\times G(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2,\omega+\omega_1+\omega_2)G(\mathbf{k}+\mathbf{k},\omega+\omega_2) \tag{C1}$$

where V_a, V_b, V_c, and V_d are vertices,

$$V_a=V(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k},\mathbf{k},\mathbf{k}_1+\mathbf{k}_2), \tag{C2}$$

$$V_b=V(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k},\mathbf{k}_1,\mathbf{k}+\mathbf{k}_2), \tag{C3}$$

$$V_c=V(\mathbf{k}_2+\mathbf{k},\mathbf{k}_2,\mathbf{k}), \tag{C4}$$

$$V_d=V(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) \tag{C5}$$

We just followed the rules of DT and integrated over all δ functions. From now on, the analyses will be parallel to that of Appendix B. Let us use Eq. (4.16) for $n(k,\omega)$ and Eq. (4.11) for $G(k,\omega)$. Now we can easily perform integration over ω_1 and ω_2. Now, as was done in Appendix B, introduce $\Sigma_b(\mathbf{k})=\Sigma_b(\mathbf{k},\omega_*)$. Since all interacting wave vectors are almost parallel, we introduce two-dimensional vectors $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ such that

$$\mathbf{k}_1=q_1\mathbf{k}/k+\boldsymbol{\kappa}_1,\quad \boldsymbol{\kappa}_1\perp\mathbf{k}, \tag{C6}$$

$$\mathbf{k}_2=q_2\mathbf{k}/k+\boldsymbol{\kappa}_2,\quad \boldsymbol{\kappa}_2\perp\mathbf{k}. \tag{C7}$$

We use $V(\mathbf{k},\mathbf{q},\mathbf{p})=A\sqrt{kqp}$. Since $\kappa_i\ll k$, we can expand resonance denominators in Eq. (C1) with respect to κ_i. The integrals will be dominated by regions where $q_i\ll k$. Putting everything together, one obtains

$$\Sigma_b(\mathbf{k})\simeq\int\frac{\pi^2 dq_1 dq_2 d\kappa_1^2 d\kappa_2^2}{2\pi^6}A^4k^3(q_1+q_2)q_1q_2\tilde{n}_{q_1}\tilde{n}_{q_2} \tag{C8}$$
$$\times\left[\left(\frac{c(\kappa_1^2+\kappa_2^2)}{2(q_1+q_2)}+\Gamma_{q_1,q_2,q_1+q_2}\right)\right.$$
$$\left.\times\left(\frac{c}{2}\frac{\kappa_1^2}{q_1}+\frac{\kappa_2^2}{q_2}+i\gamma_k\right)\left(\frac{c\kappa_2^2}{2q_2}+i\gamma_k\right)\right]^{-1}. \tag{C9}$$

Substituting $\tilde{n}_q=n/q^{-9/2}$ we see that, indeed, the dominant part comes from the region of small q_i. We can estimate all these integrals to obtain

$$\Sigma_b\simeq\frac{A^4k^3L^2n^2}{c^2\gamma_k}, \tag{C10}$$

where we used the small q cutoff $1/L$. Finally,

$$\frac{\Sigma_b}{\gamma_k}\simeq\frac{k^3L^2n^2}{\rho_0^2\gamma_k^2}\simeq\frac{1}{kL}\ll 1, \tag{C11}$$

and we conclude that the contribution from diagrams of type (b) in Fig. 1(b) is much less than the contribution from one-loop diagrams. But this is not the end of the story. Let us try to estimate contributions from diagrams of type (e) on Fig. 1(b). Following the same guidelines, we obtain

$$\Sigma_e(\mathbf{k}) = \int \frac{d\mathbf{k}_1\mathbf{k}_2}{(2\pi)^8} A^4 k k_1 k_2 (k+k_1)(k+k_1+k_2) \times (k+k_2)\tilde{n}_{k_1}\tilde{n}_{k_2}[(\omega_k+\omega_{k_1}-\omega_{k+k_1}+i\Gamma_{k,k_1,k+k_1}) \times (\omega_k+\omega_{k_1}+\omega_{k_2}-\omega_{k+k_1+k_2}+i\Gamma_{k,k_1,k_2,k+k_1+k_2}) \times (\omega_k+\omega_{k_2}-\omega_{k+k_2}+i\Gamma_{k,k_1,k+k_2})]^{-1}.$$

Let us again introduce κ_1 and κ_2 as above, and, substituting $\tilde{n}_q = n/q^{-9/2}$, we obtain the following estimation:

$$\frac{\Sigma_e}{\gamma_k} \simeq \frac{A^4 k^4 n^2 L^3}{\gamma_k^2 c^2} \simeq 1. \tag{C12}$$

Therefore we conclude that the contribution from two-loop diagrams is dominated by planar diagrams, and is of the order of the one-loop diagram contribution.

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