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Five-wave interaction on the surface of deep fluid

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Abstract

This article deals with the studying of the interaction of gravity waves propagating on the surface of an ideal fluid of infinite depth. The system of the corresponding equation is proven to be integrable up to the fourth order in power of steepness of the waves, but to be nonintegrable in the next, fifth, order. An exact formula for the five-wave scattering matrix element is obtained using diagram technique on the resonant surface. The stationary solutions of the five-wave kinetic equation are studied as well.

1. Introduction

In this article we study interaction of gravity waves propagating in one direction on the surface of an ideal fluid of infinite depth. The problem is of a big theoretical and practical importance. It is known from experiment that the distribution function of wave energy even in the active zone of a storm is almost one-dimensional in the energy-containing domain. Even more this is correct for the "swell" far away from the active zone. The point of common belief is that the main mechanism of wave interaction is four-wave scattering, satisfying the following resonant conditions

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \quad \boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3 \tag{1.1}$$

here $\boldsymbol{k}_i$ are the wave vectors of the interacting waves, and $\omega_k = \sqrt{gk}$ is the dispersion law. The corresponding effective Hamiltonian has the form

$$\mathcal{H} = \int \omega_k a_k a_k^* dk + \frac{1}{4} \int T^{kk_1}_{k_2k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots \tag{1.2}$$

(a_k are complex amplitudes of propagating waves [1–3]) and the corresponding kinetic equation is

$$\frac{\partial n}{\partial t} = \pi \int |T^{kk_1}_{k_2k_3}|^2 \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} \{n_{k_2} n_{k_3} (n_k + n_{k_1}) - n_k n_{k_1} (n_{k_2} n_{k_3})\} dk_1 dk_2 dk_3 \tag{1.3}$$

This equation is exactly equivalent (see [4]) to Hasselmann's equation, derived first in 1962 [5]. Eq. (1.3) is entirely adequate for the situation when $\boldsymbol{k}_i$ are two-dimensional vectors. But it completely fails in the one-dimensional case.

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Eqs. (1.1) have in the one-dimensional case two types of solutions:

1. The trivial solutions:

$$k_2 = k_1, \quad k_3 = k, \qquad \text{or} \qquad k_3 = k_1, \quad k_2 = k. \tag{1.4}$$

Here k and k_1 can have same or opposite signs.

2. The nontrivial solutions ($k_2 \neq k, k_1$).

These solutions exist only if the products kk_1 and k_2k_3 have opposite signs. They can be described analytically as follows. Let $k > 0$, $k_1 > 0$, $k_2 < 0$, $k_3 > 0$. Then

$$k = a(1+\zeta)^2, \quad k_1 = a(1+\zeta)^2\zeta^2, \quad k_2 = -a\zeta^2, \quad k_3 = a(1+\zeta+\zeta^2)^2 \tag{1.5}$$

If one of the conditions (1.4) holds, the expression

$$n_{k_2}n_{k_3}(n_k + n_{k_1}) - n_k n_{k_1}(n_{k_2}n_{k_3})$$

is equal to zero. So, the trivial solutions do not put any contribution to the kinetic equation (1.3). This is irrelevant if all the wave numbers have the same sign (waves propagate in the same direction). But even for waves propagating in the opposite directions, four-wave interaction vanishes. As it was shown by Dyachenko and Zakharov [6], the coefficient $T^{kk_1}_{k_2k_3}$ is identically equal to zero

$$T^{kk_1}_{k_2k_3} \equiv 0$$

on the manifold (1.5).

This remarkable identity means that the system (1.2) is approximately integrable and the kinetic equation appears for the next order only

$$\frac{\partial n}{\partial t} = st(n,n,n,n).$$

Here $st(n,n,n,n)$ is the collision term due to five-wave interaction, which is governed by the following resonant conditions

$$\omega_k + \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}, \quad k + k_1 + k_2 = k_3 + k_4 \tag{1.6}$$

The corresponding Hamiltonian has the form

$$\mathcal{H} = \int \omega_k a_k a_k^* dk + \frac{1}{12}\int T^{kk_1k_2}_{k_3k_4}\{a_k^* a_{k_1}^* a_{k_2}^* a_{k_3}a_{k_4} + c.c.\}\delta_{k+k_1+k_2-k_3-k_4}dkdk_1dk_2dk_3dk_4 + \ldots$$

The expression $st(n,n,n,n)$ looks like

$$\begin{aligned} st(n,n,n,n) &= \frac{\pi}{3}\int |T^{k_1k_2k_3}_{kk_5}|^2 f_{k_1k_2k_3kk_5}dk_1dk_2dk_3dk_5 - \frac{\pi}{2}\int |T^{kk_2k_3}_{k_4k_5}|^2 f_{kk_2k_3k_4k_5}dk_2dk_3dk_4dk_5, \\ f_{k_1k_2k_3k_4k_5} &= \delta_{k_1+k_2+k_3-k_4-k_5}\delta_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-\omega_{k_4}-\omega_{k_5}} \\ &\quad \times \{n_{k_1}n_{k_2}n_{k_3}(n_{k_4}+n_{k_5}) - n_{k_4}n_{k_5}(n_{k_1}n_{k_2} + n_{k_1}n_{k_3} + n_{k_2}n_{k_3})\} \end{aligned} \tag{1.7}$$

The expression (1.7) was found by Krasitskii [7]. He also found "in principle" the expression for $T^{kk_1k_2}_{k_3k_4}$. But his final formula is extraordinarily complicated and cumbersome and can hardly be used for any practical purpose. He used the technique of canonical transformation which excludes gradually the low order nonlinear terms in the Hamiltonian.

In this article we evaluate the coefficient $T^{kk_1k_2}_{k_3k_4}$ on the resonant surface (1.6). Our final formulae are astonishingly simple. This is one more miracle in the theory of surface waves. We also find Kolmogorov's solution of the stationary equation

$$st(n,n,n,n) = 0$$

We will use a technique different from that used in [1–3,5].

2. Conformal canonical variables

The basic set of equations describing a two-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is the following one:

$$\phi_{xx} + \phi_{zz} = 0 \quad (\phi_z \to 0, z \to -\infty),$$
$$\eta_t + \eta_x\phi_x = \phi_z\Big|_{z=\eta}, \quad \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0\Big|_{z=\eta};$$

here $\eta(x,t)$ is the shape of a surface, $\phi(x,z,t)$ is a potential function of the flow and g is a gravitational constant. As was shown by Zakharov in [1], the potential on the surface $\psi(x,t) = \phi(x,z,t)|_{z=\eta}$ and $\eta(x,t)$ are canonically conjugated, and their Fourier transforms satisfy the equations

$$\frac{\partial\psi_k}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta_k^*}, \quad \frac{\partial\eta_k}{\partial t} = \frac{\delta\mathcal{H}}{\delta\psi_k^*}.$$

Here $\mathcal{H} = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz, \quad U = \frac{g}{2}\int \eta^2 dx$$

A Hamiltonian can be expanded in an infinite series in powers of a characteristic wave steepness $k\eta_k << 1$ (see [1,2]) by using an iterative procedure. All terms up to the fifth order of this series contribute to the amplitude of five-wave interaction.

Here we prefer to do that performing first a certain canonical transformation from the variables ψ, η to new canonical variables.

Let us perform, following Kuznetsov, Spector and Zakharov [8], a conformal mapping of the domain $z < \eta(x,t)$ to the lower half-plane of the complex variable $w = u + iv$, $-\infty < u < \infty$, $-\infty < v < 0$. The shape of the surface is parametrized by two functions

$$z(u,t), \quad x(u,t)$$

which are connected by the Hilbert transformation

$$x(u,t) = u - \hat{H}(z(u,t)), \quad \hat{H}(f(u)) = \frac{1}{\pi}P.V.\int \frac{f(u')du'}{u'-u}$$

We introduce also the complex velocity potential

$$\Phi(w,t) = \Psi(u,v,t) + i\Theta(u,v,t)$$

On the surface ($v = 0$)

$$\Theta(u,0,t) = \hat{H}(\Psi(u,0,t)),$$

New canonical variables can be obtained using the variational principle for the action. With the old variables the action is

$$S = \int dt \left\{ \int \psi(x,t)\eta_t(x,t)dx - \mathcal{H} \right\}.$$

After conformal mapping [8] it acquires the form:

$$S = \int L dt, \quad L = \int \{\Psi(z_t x_u - x_t z_u) + \tfrac{1}{2}\Psi\hat{H}\Psi_u - \tfrac{1}{2}gz^2 x_u\} du.$$

The Lagrangian function L can be rewritten as

$$L = \int \{z_t(\Psi x_u - \hat{H}(\Psi z_u)) + \tfrac{1}{2}\Psi\hat{H}\Psi_u - \tfrac{1}{2}gy^2 x_u\} du.$$

and the new canonical variables are $z(u,t)$ and $\mathcal{P}(u,t) = \Psi x_u - \hat{H}(\Psi z_u)$. Ψ can be easily inverted as

$$\Psi = \frac{\mathcal{P}x_u + \hat{H}(\mathcal{P}z_u)}{x_u^2 + z_u^2} \tag{2.1}$$

and the Hamiltonian of the system is

$$\mathcal{H} = \frac{1}{2}\int \{gz^2 x_u - \Psi\hat{H}\Psi_u\} du$$

where Ψ is equal to (2.1). The equations of motion can be written in the explicit Hamiltonian form which includes the integral Hilbert's operator:

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta z}, \quad \frac{\partial z}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathcal{P}}$$

3. Perturbation expansion for the Hamiltonian

One can introduce a new variable $\tilde{x}$ as

$$x = u + \tilde{x}, \quad x_u = 1 + \tilde{x}_u, \quad \tilde{x} = -\hat{H}z$$

Then

$$\Psi = \frac{\mathcal{P} + \mathcal{P}\tilde{x}_u + \hat{H}(\mathcal{P}z_u)}{(1+\tilde{x}_u)^2 + z_u^2}$$

Now one can expand Ψ in powers of $\tilde{x}_u$ and z_u

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)} + \Psi^{(4)} + \ldots,$$
$$\Psi^{(1)} = \mathcal{P}, \quad \Psi^{(2)} = \hat{H}(z_u\mathcal{P}) - \tilde{x}_u\mathcal{P},$$
$$\Psi^{(3)} = \mathcal{P}(\tilde{x}_u^2 - z_u^2) - 2\tilde{x}_u\hat{H}(z_u\mathcal{P}), \quad \Psi^{(4)} = \mathcal{P}\tilde{x}_u(3z_u^2 - \tilde{x}_u^2) + (3\tilde{x}_u^2 - z_u^2)\hat{H}(z_u\mathcal{P})$$

The Hamiltonian of the system is

$$\mathcal{H} = \frac{1}{2}\int \{gz^2(1+\tilde{x}_u) - \Psi\hat{H}\Psi_u\}du$$

Now $\mathcal{H}$ can be expanded as follows

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \ldots \tag{3.1}$$

here

$$\mathcal{H}_2 = \frac{1}{2}\int (gz^2 - \Psi^{(1)}\hat{H}\Psi_u^{(1)})du, \quad \mathcal{H}_3 = \frac{1}{2}\int (gz^2\tilde{x}_u - 2\Psi^{(2)}\hat{H}\Psi_u^{(1)})du,$$
$$\mathcal{H}_4 = -\frac{1}{2}\int (2\Psi^{(3)}\hat{H}\Psi_u^{(1)} + \Psi^{(2)}\hat{H}\Psi_u^{(2)})du, \quad \mathcal{H}_5 = -\int (\Psi^{(4)}\hat{H}\Psi_u^{(1)} + \Psi^{(3)}\hat{H}\Psi_u^{(2)})du$$

Let's introduce the Fourier transform:

$$f_k = \frac{1}{\sqrt{2\pi}}\int f(u)e^{-iku}du, \quad f(u) = \frac{1}{\sqrt{2\pi}}\int f_k e^{iku}dk$$

After simple, but somewhat tedious calculations one can find

$$\mathcal{H}_2 = \frac{1}{2}\int (g|z_k|^2 + |k||\mathcal{P}_k|^2)dk,$$
$$\mathcal{H}_3 = \frac{1}{2\sqrt{2\pi}}\int z_{k_1}\{S_{k_1k_2k_3}z_{k_2}z_{k_3} - F^{k_1}_{k_2k_3}\mathcal{P}_{k_2}\mathcal{P}_{k_3}\}\delta_{k_1+k_2+k_3}dk_1dk_2dk_3,$$
$$\mathcal{H}_4 = \frac{1}{4\pi}\int M^{k_1k_2}_{k_3k_4}z_{k_1}z_{k_2}\mathcal{P}_{k_3}\mathcal{P}_{k_4}\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4,$$
$$\mathcal{H}_5 = \frac{1}{2(2\pi)^{\frac{3}{2}}}\int N^{k_1k_2k_3}_{k_4k_5}z_{k_1}z_{k_2}z_{k_3}\mathcal{P}_{k_4}\mathcal{P}_{k_5}\delta_{k_1+k_2+k_3+k_4+k_5}dk_1dk_2dk_3dk_4dk_5$$

Here $S_{k_1k_2k_3}$, $F^{k_1}_{k_2k_3}$, $M^{k_1k_2}_{k_3k_4}$ and $N^{k_1k_2k_3}_{k_4k_5}$ are the functions symmetric in upper and lower groups of indices. Namely

$$S_{k_1k_2k_3} = \frac{g}{3}(|k_1| + |k_2| + |k_3|), \quad L_{k_1k_2} = k_1k_2 + |k_1k_2|$$
$$F^{k_1}_{k_2k_3} = L_{-k_1k_2} + L_{-k_1k_3} = |k_1|(|k_1| + |k_2| + |k_3|) \tag{3.2}$$

The expression for $M^{k_1k_2}_{k_3k_4}$ and $N^{k_1k_2k_3}_{k_4k_5}$ are given in Appendix A.

It is convenient to introduce a normal complex variable a_k

$$y_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}), \quad \mathcal{P}_k = -i\sqrt{\frac{2g}{\omega_k}}(a_k - a^*_{-k})$$

which satisfies the equation of motion

$$\frac{\partial a_k}{\partial t} + i\frac{\delta\mathcal{H}}{\delta a^*_k} = 0.$$

here $\omega_k = \sqrt{g|k|}$ is the dispersion law for the gravity waves.

In the normal variable a_k the second order term in the Hamiltonian acquires the form:

$$\mathcal{H}_2 = \int \omega_k a_k a^*_k dk$$

The third order term is:

$$\mathcal{H}_3 = \frac{1}{2}\int V^{k_1}_{k_2k_3}\{a^*_{k_1}a_{k_2}a_{k_3} + a_{k_1}a^*_{k_2}a^*_{k_3}\}\delta_{k_1-k_2-k_3}dk_1dk_2dk_3$$
$$+\frac{1}{6}\int U_{k_1k_2k_3}\{a_{k_1}a_{k_2}a_{k_3} + a^*_{k_1}a^*_{k_2}a^*_{k_3}\}\delta_{k_1+k_2+k_3}dk_1dk_2dk_3 \tag{3.3}$$

$$V^{k_1}_{k_2k_3} = \frac{g^{\frac{1}{4}}}{2\sqrt{4\pi}}(|k_1|+|k_2|+|k_3|)\left(|k_1k_2k_3|^{\frac{1}{4}} + \left(\frac{k_1}{k_2k_3}\right)^{\frac{1}{4}}|k_1| - \left(\frac{k_2}{k_1k_3}\right)^{\frac{1}{4}}|k_2| - \left(\frac{k_3}{k_1k_2}\right)^{\frac{1}{4}}|k_3|\right)$$
$$U_{k_1k_2k_3} = \frac{g^{\frac{1}{4}}}{2\sqrt{4\pi}}(|k_1|+|k_2|+|k_3|)\left(|k_1k_2k_3|^{\frac{1}{4}} + \left(\frac{k_1}{k_2k_3}\right)^{\frac{1}{4}}|k_1| + \left(\frac{k_2}{k_1k_3}\right)^{\frac{1}{4}}|k_2| + \left(\frac{k_3}{k_1k_2}\right)^{\frac{1}{4}}|k_3|\right)$$

The fourth order term in the Hamiltonian consists of three terms:

$$\mathcal{H}_4 = \mathcal{H}_4^{4\Leftrightarrow 0} + \mathcal{H}_4^{3\Leftrightarrow 1} + \mathcal{H}_4^{2\Leftrightarrow 2}$$

describing different types of the wave-wave interactions. The term corresponding to the $4 \Leftrightarrow 0$ interaction has the form:

$$\mathcal{H}_4^{4\Leftrightarrow 0} = \frac{1}{24}\int R_{k_1k_2k_3k_4}(a_{k_1}a_{k_2}a_{k_3}a_{k_4} + a^*_{k_1}a^*_{k_2}a^*_{k_3}a^*_{k_4})\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4$$

where $R_{k_1k_2k_3k_4}$ is:

$$R_{k_1k_2k_3k_4} = \frac{-1}{4\pi}\left(\left|\frac{k_1k_2}{k_3k_4}\right|^{\frac{1}{4}}M^{k_1k_2}_{k_3k_4} + \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}}M^{k_1k_3}_{k_2k_4} + \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}}M^{k_1k_4}_{k_2k_3}\right.$$
$$\left.+\left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}}M^{k_2k_3}_{k_1k_4} + \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}}M^{k_2k_4}_{k_1k_3} + \left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}}M^{k_3k_4}_{k_1k_2}\right) \tag{3.4}$$

The term corresponding to the $3 \Leftrightarrow 1$ interaction has the form:

$$\mathcal{H}_4^{3\Leftrightarrow 1} = \frac{1}{6}\int G^{k_1}_{k_2k_3k_4}(a^*_{k_1}a_{k_2}a_{k_3}a_{k_4} + a_{k_1}a^*_{k_2}a^*_{k_3}a^*_{k_4})\delta_{k_1-k_2-k_3-k_4}dk_1dk_2dk_3dk_4$$

where $G^{k_1}_{k_2k_3k_4}$:

$$G^{k_1}_{k_2k_3k_4} = \frac{-1}{4\pi}\left(\left|\frac{k_1k_2}{k_3k_4}\right|^{\frac{1}{4}}M^{-k_1k_2}_{k_3k_4} + \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}}M^{-k_1k_3}_{k_2k_4} + \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}}M^{-k_1k_4}_{k_2k_3}\right.$$
$$\left.-\left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}}M^{k_2k_3}_{-k_1k_4} - \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}}M^{k_2k_4}_{-k_1k_3} - \left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}}M^{k_3k_4}_{-k_1k_2}\right) \tag{3.5}$$

The term corresponding to the $2 \Leftrightarrow 2$ interaction has the form:

$$\mathcal{H}_4^{2\Leftrightarrow 2} = \frac{1}{4}\int W^{k_1k_2}_{k_3k_4}a^*_{k_1}a^*_{k_2}a_{k_3}a_{k_4}\delta_{k_1+k_2-k_3-k_4}dk_1dk_2dk_3dk_4$$

where $W^{k_1k_2}_{k_3k_4}$:

$$W^{k_1k_2}_{k_3k_4} = \frac{-1}{4\pi}\left(\left|\frac{k_1k_2}{k_3k_4}\right|^{\frac{1}{4}}M^{-k_1-k_2}_{k_3k_4} - \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}}M^{-k_1k_3}_{-k_2k_4} - \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}}M^{-k_1k_4}_{k_3-k_2}\right.$$
$$\left.-\left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}}M^{-k_2k_3}_{-k_1k_4} - \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}}M^{-k_2k_4}_{-k_1k_3} + \left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}}M^{k_3k_4}_{-k_1-k_2}\right) \tag{3.6}$$

Among the different terms of the fifth order we consider only the term corresponding to the process (1.6):

$$\mathcal{H}_5 = \frac{1}{12}\int Q^{k_1k_2k_3}_{k_4k_5}\{a^*_{k_1}a^*_{k_2}a^*_{k_3}a_{k_4}a_{k_5} + c.c.\}\delta_{k_1+k_2+k_3-k_4-k_5}dk_1dk_2dk_3dk_4dk_5$$

where

$$\begin{aligned}Q^{k_1k_2k_3}_{k_4k_5} = \frac{-3}{(4\pi)^{\frac{3}{2}}g^{\frac{1}{4}}}\Bigg\{&+\left(\frac{|k_1k_2k_3|}{|k_4k_5|}\right)^{\frac{1}{4}}N^{-k_1,-k_2,-k_3}_{k_4,k_5} - \left(\frac{|k_1k_2k_4|}{|k_3k_5|}\right)^{\frac{1}{4}}N^{-k_1,-k_2,k_4}_{-k_3,k_5} - \left(\frac{|k_1k_2k_5|}{|k_3k_4|}\right)^{\frac{1}{4}}N^{-k_1,-k_2,k_5}_{k_4,-k_3}\\ &- \left(\frac{|k_1k_3k_4|}{|k_2k_5|}\right)^{\frac{1}{4}}N^{-k_1,k_4,-k_3}_{-k_2,k_5} - \left(\frac{|k_1k_3k_5|}{|k_2k_4|}\right)^{\frac{1}{4}}N^{-k_1,k_5,-k_3}_{k_4,-k_2} - \left(\frac{|k_2k_3k_4|}{|k_1k_5|}\right)^{\frac{1}{4}}N^{k_4,-k_2,-k_3}_{-k_1,k_5}\\ &- \left(\frac{|k_2k_3k_5|}{|k_1k_4|}\right)^{\frac{1}{4}}N^{k_5,-k_2,-k_3}_{k_4,-k_1} + \left(\frac{|k_1k_4k_5|}{|k_2k_3|}\right)^{\frac{1}{4}}N^{-k_1,k_4,k_5}_{-k_2,-k_3}\\ &+ \left(\frac{|k_2k_4k_5|}{|k_1k_3|}\right)^{\frac{1}{4}}N^{k_4,-k_2,k_5}_{-k_1,-k_3} + \left(\frac{|k_3k_4k_5|}{|k_1k_2|}\right)^{\frac{1}{4}}N^{k_4,k_5,-k_3}_{-k_1,-k_2}\Bigg\}\end{aligned}$$

4. Effective four-wave Hamiltonian

The Hamiltonian $\mathcal{H}$ in the normal variables a_k is too complicated to work with. Our purpose is to simplify the Hamiltonian to the form:

$$\begin{aligned}\mathcal{H} = &\int \omega_k b_k b^*_k dk + \frac{1}{4}\int T^{k_1k_2}_{k_3k_4}b^*_{k_1}b^*_{k_2}b_{k_3}b_{k_4}\delta_{k_1+k_2-k_3-k_4}dk_1dk_2dk_3dk_4\\ &+\frac{1}{12}\int T^{k_1k_2k_3}_{k_4k_5}\{b^*_{k_1}b^*_{k_2}b^*_{k_3}b_{k_4}b_{k_5} + c.c.\}\delta_{k_1+k_2+k_3-k_4-k_5}dk_1dk_2dk_3dk_4dk_5\end{aligned} \tag{4.1}$$

To do that we have to perform a transformation (Zakharov, 1975 [9], Krasitskii, 1990 [10])

$$\begin{aligned}a_k = b_k &+ \int \Gamma^k_{k_1k_2}b_{k_1}b_{k_2}\delta_{k-k_1-k_2} - 2\int \Gamma^{k_2}_{kk_1}b^*_{k_1}b_{k_2}\delta_{k+k_1-k_2}\\ &+ \int \Gamma_{kk_1k_2}b^*_{k_1}b^*_{k_2}\delta_{k+k_1+k_2} + \int B^{kk_1}_{k_2k_3}b^*_{k_1}b_{k_2}b_{k_3}\delta_{k+k_1-k_2-k_3} + \dots\end{aligned} \tag{4.2}$$

$$\Gamma^k_{k_1k_2} = -\frac{1}{2}\frac{V^k_{k_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \qquad \Gamma_{kk_1k_2} = -\frac{1}{2}\frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}$$

$$\begin{aligned}B^{kk_1}_{k_2k_3} = &\Gamma^{k_1}_{k_2k_1-k_2}\Gamma^{k_3}_{kk_3-k} + \Gamma^{k_1}_{k_3k_1-k_3}\Gamma^{k_2}_{kk_2-k} - \Gamma^{k}_{k_2k-k_2}\Gamma^{k_3}_{k_1k_3-k_1} - \Gamma^{k_1}_{k_3k_1-k_3}\Gamma^{k_2}_{k_1k_2-k_1}\\ &-\Gamma^{k+k_1}_{kk_1}\Gamma^{k_2+k_3}_{k_2k_3} + \Gamma_{-k_1-k_1kk_1}\Gamma_{-k_2-k_3k_2k_3} + \tilde{B}^{kk_1}_{k_2k_3}\end{aligned}$$

Here $\tilde{B}^{kk_1}_{k_2k_3}$ is an arbitrary function satisfying the conditions:

$$\tilde{B}^{kk_1}_{k_2k_3} = \tilde{B}^{k_1k}_{k_2k_3} = \tilde{B}^{kk_1}_{k_3k_2} = -(\tilde{B}^{k_2k_3}_{kk_1})^*$$

The transformation (4.2) is canonical up to terms of the order of $|b_k|^4$. It excludes from the Hamiltonian all cubic terms. The form of the $T^{kk_1}_{k_2k_3}$ depends on the choice of the function $\tilde{B}^{kk_1}_{k_2k_3}$. Let first $\tilde{B}^{kk_1}_{k_2k_3} = 0$. Then $T^{kk_1}_{k_2k_3} = \hat{T}^{kk_1}_{k_2k_3}$ and

$$\begin{aligned}\hat{T}^{kk_1}_{k_2k_3} = W^{k_1k}_{k_2k_3} &- \frac{1}{2}V^{k}_{k_2k-k_2}V^{k_3}_{k_1k_3-k_1}\left[\frac{1}{\omega_{k_2}+\omega_{k-k_2}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_3-k_1}-\omega_{k_3}}\right]\\ &-\frac{1}{2}V^{k_1}_{k_2k_1-k_2}V^{k_3}_{kk_3-k}\left[\frac{1}{\omega_{k_2}+\omega_{k_1-k_2}-\omega_{k_1}}+\frac{1}{\omega_k+\omega_{k_3-k}-\omega_{k_3}}\right]\\ &-\frac{1}{2}V^{k}_{k_3k-k_3}V^{k_2}_{k_1k_2-k_1}\left[\frac{1}{\omega_{k_3}+\omega_{k-k_3}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_2-k_1}-\omega_{k_2}}\right]\\ &-\frac{1}{2}V^{k_1}_{k_3k_1-k_3}V^{k_2}_{kk_2-k}\left[\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}}+\frac{1}{\omega_k+\omega_{k_2-k}-\omega_{k_2}}\right]\\ &-\frac{1}{2}V^{k+k_1}_{kk_1}V^{k_2+k_3}_{k_2k_3}\left[\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right]\\ &-\frac{1}{2}U_{-k-k_1kk_1}U_{-k_2-k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right]\end{aligned} \tag{4.3}$$

The expression (4.3) in spite of its complexity has some remarkable properties. Let us consider all nontrivial solutions of Eqs. (1.1). They consist of the manifold (1.5) and of seven other manifolds which are obtained from (1.5) by the permutations

$$k \leftrightarrow k_1, \quad k_2 \leftrightarrow k_3, \quad (k, k_1) \leftrightarrow (k_2, k_3)$$

Direct analytic calculation shows that

$$\hat{T}^{kk_1}_{k_2k_3} \equiv 0$$

on all these manifolds. Recently Craig and Worfolk [14] confirmed this cancellation by an independent calculation. Another remarkable feature of $\hat{T}^{kk_1}_{k_2k_3}$ is the simplicity of its diagonal part. Let us denote

$$T_{kk_1} = \hat{T}^{kk_1}_{kk_1}$$

A simple, but long calculation ([6,11]) shows that

$$T_{kk_1} = \frac{1}{4\pi^2}kk_1 \min\{|k|, |k_1|\}$$

Let us consider the function

$$\tilde{T}^{kk_1}_{k_2k_3} = [\tfrac{1}{2}(T_{kk_2}+T_{kk_3}+T_{k_1k_2}+T_{k_1k_3}) - \tfrac{1}{4}(T_{kk}+T_{k_1k_1}+T_{k_2k_2}+T_{k_3k_3})]\theta(kk_1k_2k_3),$$

$$\theta(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Obviously

$$\tilde{T}^{kk_1}_{kk_1} = \hat{T}^{kk_1}_{kk_1} = T_{kk_1}$$

Let us choose

$$\tilde{B}^{kk_1}_{k_2k_3} = \frac{\tilde{T}^{kk_1}_{k_2k_3} - \hat{T}^{kk_1}_{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}$$

One can check that this transformation replaces $\hat{T}^{kk_1}_{k_2k_3}$ by $\tilde{T}^{kk_1}_{k_2k_3}$. So, one can assume in future that

$$T^{kk_1}_{k_2k_3} = \tilde{T}^{kk_1}_{k_2k_3}$$

In the case of periodic boundary conditions

$$b_k = \sum_{n=-\infty}^{\infty} b_n \delta_{k-k_0 n}$$

Introducing the notation

$$\omega_{k_0 n} = \omega_n, \quad T^{k_0 n k_0 n_1}_{k_0 n_2 k_0 n_3} = T^{n n_1}_{n_2 n_3}$$

we can find that now

$$\frac{\partial b_n}{\partial t} + i\frac{\delta \mathcal{H}}{\delta b_n^*} = 0.$$

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \omega_n |b_n|^2 + \frac{1}{4} \sum_{n,n_1,n_2,n_3=-\infty}^{\infty} T^{n n_1}_{n_2 n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} \delta_{n+n_1,n_2+n_3}$$

In the canonical transformation (4.2) all the integrals are replaced now by discrete sums. In particular instead of $\tilde{B}^{k k_1}_{k_2 k_3}$ we have now

$$\tilde{B}^{n n_1}_{n_2 n_3} = \tilde{B}^{k_0 n k_0 n_1}_{k_0 n_2 k_0 n_3}$$

Let us choose

$$\tilde{B}^{n n_1}_{n_2 n_3} = \frac{T^{n n_1}_{n_2 n_3} - T_{n n_1}\delta_{n,n_2}\delta_{n_1,n_3} - T_{n n_1}\delta_{n,n_3}\delta_{n_1,n_2}}{\omega_n + \omega_{n_1} - \omega_{n_2} - \omega_{n_3}} \tag{4.4}$$

This expression has no singularities on the diagonals $n_2 = n, n_3 = n_1$ and $n_2 = n_1, n_3 = n$. The transformation (4.2) with (4.4) brings the Hamiltonian to Birghoff's form

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \omega_n |b_n|^2 + \frac{1}{4} \sum_{n,n_1=-\infty}^{\infty} T_{n n_1} |b_n|^2 |b_{n_1}|^2 \tag{4.5}$$

This is a Hamiltonian of an integrable system. A level of nonlinearity, allowing the representation (4.5), has to be studied separately.

5. Five-wave interaction on resonant surface

To find $T^{k k_1 k_2}_{k_3 k_4}$ one can calculate the terms of the order of b^3 and b^4 in the canonical transformation (4.2). This very cumbersome procedure was fulfilled by V. Krasitskii [7]. This is a kind of feat, but the resulting formulae are so complicated that they can hardly be used for any practical purpose. In this article we offer a much more simple and clear way to $T^{k k_1 k_2}_{k_3 k_4}$ involving a Feynman diagram technique for the scattering matrix.

Our intermediate formulae are very complicated also, due to one-to-one correspondence of each term in the expression to a certain graphic picture, however all the procedure is very easily controlled. It is very remarkable that our final formula is very simple.

First we introduce so called formal classical scattering matrix. Let

$$H = H_2 + H_{\text{int}}$$

be a Hamiltonian of a some nonlinear system in a homogeneous space. Here $H_2 = \int \omega_k a_k a_k^* dk$, and H_{int} is in general case an infinite series in power a_k, a_k^*

$$H_{\rm int} = H_3 + H_4 + \dots$$

The motion equation is as usual

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0. \tag{5.1}$$

One can change H to the auxiliary Hamiltonian

$$\hat{H} = H_2 + e^{-\epsilon|t|}H_{\rm int}$$

Now Eq. (5.1) becomes linear at $t \to \pm\infty$ and

$$a_k(t) \to c_k^{\pm} e^{-i\omega_k t}, \quad t \to \pm\infty$$

The asymptotic states $c_k^{\pm}$ are not independent, and actually

$$c_k^+ = \hat{S}_\epsilon[c_k^-]$$

$\hat{S}_\epsilon[c_k^-]$ is a nonlinear operator which can be presented as a series in powers of c^-, c^{-*}. We will treat this series as formal one and will not care about its convergence. A formal series which is a result of the limiting transition

$$\hat{S}[c_k^-] = \lim_{\epsilon\to 0} \hat{S}_\epsilon[c_k^-]$$

is the formal classic scattering matrix. It has the following form

$$\begin{aligned}\hat{S}[c_k^-] = c_k^- - \sum_{n+m\geq 3} \frac{2\pi i}{(n-1)!m!} \int S_{nm}(k, k_1, \dots, k_{n-1}; k_n, \dots, k_{n+m-1}) \\ \times \delta_{k+k_1+\dots+k_{n-1}-k_n-\dots-k_{n+m-1}} \delta_{\omega_k+\omega_{k_1}+\dots+\omega_{k_{n-1}}-\omega_{k_n}-\dots-\omega_{k_{n+m-1}}} \\ \times c^{-*}_{k_1} \dots c^{-*}_{k_{n-1}} c^-_{k_n} \dots c^-_{k_{n+m-1}} dk_1 \dots dk_{n+m-1}\end{aligned} \tag{5.2}$$

The functions S_{nm} are the elements of the scattering matrix. They are defined on the resonant manifolds

$$k + k_1 + \dots + k_{n-1} = k_n + \dots + k_{n+m-1}, \quad \omega_k + \omega_{k_1} + \dots + \omega_{k_{n-1}} = \omega_{k_n} + \dots + \omega_{k_{n+m-1}} \tag{5.3}$$

Two basic properties of the matrix elements are important for us.

(i) The value of the matrix element S_{nm} on the resonant manifold (5.3) is invariant with respect to canonical transformation (4.2).

(ii) There is a simple algorithm for calculation of the matrix elements. The element S_{nm} is a finite sum of terms which can be expressed through the coefficients of the Hamiltonians $H_i, i \leq n+m$. Each term can be marked by a certain Feynman diagram, having no internal loops. The rules of correspondence are described in Appendix B.

Actually the classical scattering matrix is nothing but the Feynman scattering matrix taken in the 'tree' approximation. This approximation makes a number of terms finite for any element.

Our idea how to find $T^{kk_1k_2}_{k_3k_4}$ is the following. We calculate first nonzero elements of the scattering matrix for the Hamiltonian (3.1) and for the Hamiltonian (4.1). Because these two Hamiltonians are connected by the canonical transformation (4.2), the results must coincide. For surface gravity waves the first nontrivial matrix element is S_{22}. In terms of the Hamiltonian (4.1) it is

$$S_{22}(k, k_1, k_2, k_3) = T^{kk_1}_{k_2k_3}$$

Being calculated for the Hamiltonian (3.1), this element consist of six terms. They are presented (together with the corresponding diagrams) in Appendix B. One can see that the result coincide with the expression (4.3) on the resonant manifold (1.1).

In the one-dimensional case the first integral in (5.2) can be calculated so that the first two terms in (5.2) has the form

$$c^+_k = c^-_k\left(1 - \pi i \int_{-\infty}^{\infty} \frac{T_{kk_1}}{|\omega_k - \omega_{k_1}|}|c^-_{k_1}|^2 dk_1\right) \tag{5.4}$$

This formula one more time stresses the fact that four-wave nonlinear processes in the one-dimensional case lead only to trivial scattering which does not produce "new wave vectors". The integral in (5.4) diverges logarithmically. That is why our scattering matrix is "formal". In reality in the one-dimensional case the waves don't become linear indeed if $t \to \infty$. They acquire a logarithmically growing phase (see Zakharov, Manakov [13]).

The first nontrivial element of the scattering matrix in the one-dimensional case is

$$S_{32}(k, k_1, k_2, k_3, k_4) = T^{kk_1k_2}_{k_3k_4}$$

Being calculated in terms of the initial Hamiltonian (3.1) it consists of 81 terms. Their expressions together with diagrams are presented in Appendix C.

In spite of the complexity of the expression for $T^{kk_1k_2}_{k_3k_4}$ it can be enormously simplified on the resonant manifold. We will discuss here only the case when all k_i in the resonant conditions

$$k + k_1 + k_2 = k_3 + k_4, \quad \omega_k + \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4} \tag{5.5}$$

have the same sign.

The manifold (5.5) can be parametrised as follows

$$\begin{aligned} &k = a(p^2 - q^2 + 1 - 2p)^2, \quad k_1 = a(p^2 - q^2 + 1 + 2p)^2, \quad k_2 = 16a, \\ &k_3 = a(p^2 - q^2 + 3 - 2q)^2, \quad k_4 = a(p^2 - q^2 + 3 + 2q)^2, \end{aligned} \tag{5.6}$$

here $0 < |p|, |q| < 1, |p \pm q| < 1, a > 0$. It is easy to see that k_i here satisfy the inequality:

$$k, k_1 < k_3, k_4 < k_2$$

Plugging the parametrization (5.6) in the expression obtained for $T^{kk_1k_2}_{k_3k_4}$ we get a sum of more than thousands terms. Using the program for analytical calculations 'Mathematica' we manage to simplify this expression to the following form

$$T^{k_1k_2k_3}_{k_4k_5} = \frac{2}{g^{1/2}\pi^{3/2}}\sqrt{\frac{\omega_{k_1}\omega_{k_2}\omega_{k_3}}{\omega_{k_4}\omega_{k_5}}}\frac{k_1k_2k_3k_4k_5}{\max(k_1, k_2, k_3)} \tag{5.7}$$

This formula is the main result of the presented article. The fact that $T^{kk_1k_2}_{k_3k_4} \neq 0$ on the resonant surface means that the system of gravity waves on a surface of deep water is a *nonintegrable Hamiltonian system.*

6. Five-wave kinetic equation and its solutions

The matrix element $T^{kk_1}_{k_2k_3k_4}$ for $3 \Leftrightarrow 2$ process along with $T^{kk_1}_{k_2k_3}$ allows us to derive kinetic equation which includes four- and five-wave interactions.

The dynamical equation for b_k (with the Hamiltonian (4.1)) is:

$$\frac{\partial b_k}{\partial t} + i\omega_k b_k + i\frac{1}{2}\int T^{kk_1}_{k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3}\, dk_1 dk_2 dk_3 + i\frac{1}{4}\int T^{kk_2k_3}_{k_4k_5} b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5} \delta_{k+k_2+k_3-k_4-k_5}\, dk_2 dk_3 dk_4 dk_5 + i\frac{1}{6}\int T^{k_1k_2k_3}_{kk_5} b_{k_1} b_{k_2} b_{k_3} b^*_{k_5} \delta_{k_1+k_2+k_3-k-k_5}\, dk_1 dk_2 dk_3 dk_5 = 0. \tag{6.1}$$

Introducing standard pair correlation function n_k

$$\langle b_k b_{k_1}\rangle = n_k \delta_{k-k_1}$$

we can derive from (6.1) the equation for n_k:

$$\frac{\partial n_k}{\partial t} = \mathrm{Im}\int T^{kk_1}_{k_2k_3}\langle b^*_k b^*_{k_1} b_{k_2} b_{k_3}\rangle \delta_{k+k_1-k_2-k_4}\, dk_1 dk_2 dk_3 + \frac{1}{2}\mathrm{Im}\int T^{kk_2k_3}_{k_4k_5}\langle b^*_k b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5}\rangle \delta_{k+k_2+k_3-k_4-k_5}\, dk_2 dk_3 dk_4 dk_5 - \frac{1}{3}\mathrm{Im}\int T^{k_1k_2k_3}_{kk_5}\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b_k b_{k_5}\rangle \delta_{k_1+k_2+k_3-k-k_5}\, dk_1 dk_2 dk_3 dk dk_5$$

It is obvious that $T^{kk_1}_{k_2k_3}$ contributes to the equation for the fourth-order correlator, while $T^{k_1k_2k_3}_{k_4k_5}$ contributes to the equation for the fifth-order correlator only (due to the fact that the seventh-order correlators vanish). The fifth-order correlation function $\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5}\rangle$ can be expressed through the eighth-order correlator:

$$\left(\frac{\partial}{\partial t} - i(\omega_{k_1} + \omega_{k_2} + \omega_{k_3} - \omega_{k_4} - \omega_{k_5})\right)\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5}\rangle = i\frac{1}{4}\int T^{k_1p_2p_3}_{p_4p_5}\langle b^*_{k_2} b^*_{k_3} b^*_{p_4} b^*_{p_5} b_{k_4} b_{k_5} b_{p_2} b_{p_3}\rangle \delta_{k_1+p_2+p_3-p_4-p_5}\, dp_2 dp_3 dp_4 dp_5 + i\frac{1}{4}\int T^{k_2p_2p_3}_{p_4p_5}\langle b^*_{k_1} b^*_{k_3} b^*_{p_4} b^*_{p_5} b_{k_4} b_{k_5} b_{p_2} b_{p_3}\rangle \delta_{k_2+p_2+p_3-p_4-p_5}\, dp_2 dp_3 dp_4 dp_5 + i\frac{1}{4}\int T^{k_3p_2p_3}_{p_4p_5}\langle b^*_{k_1} b^*_{k_2} b^*_{p_4} b^*_{p_5} b_{k_4} b_{k_5} b_{p_2} b_{p_3}\rangle \delta_{k_3+p_2+p_3-p_4-p_5}\, dp_2 dp_3 dp_4 dp_5 - i\frac{1}{6}\int T^{p_1p_2p_3}_{k_4p_5}\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b^*_{p_5} b_{k_5} b_{p_1} b_{p_2} b_{p_3}\rangle \delta_{p_1+p_2+p_3-k_4-p_5}\, dp_1 dp_2 dp_3 dp_5 - i\frac{1}{6}\int T^{p_1p_2p_3}_{k_5p_5}\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b^*_{p_5} b_{k_4} b_{p_1} b_{p_2} b_{p_3}\rangle \delta_{p_1+p_2+p_3-k_5-p_5}\, dp_1 dp_2 dp_3 dp_5$$

Applying random phase approximation for the eighth-order correlator (to split it in a product of pair correlation functions) and assuming slow variation in time for the fifth-order correlator, one can get the following expression for $\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5}\rangle$:

$$\mathrm{Im}\langle b^*_{k_1} b^*_{k_2} b^*_{k_3} b_{k_4} b_{k_5}\rangle = \pi T^{k_1k_2k_3}_{k_4k_5}\delta_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-\omega_{k_4}-\omega_{k_5}} \times\{n_{k_1}n_{k_2}n_{k_3}(n_{k_4}+n_{k_5}) - n_{k_4}n_{k_5}(n_{k_1}n_{k_2}+n_{k_1}n_{k_3}+n_{k_2}n_{k_3})\} \tag{6.2}$$

In (6.2) we drop the terms which are out of the resonant surface. For the fourth-order correlator we have the following equaton [15]:

$$\mathrm{Im}\langle b_k^* b_{k_1}^* b_{k_2} b_{k_3}\rangle = \pi T_{k_2 k_3}^{k k_1} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} \{n_{k_2} n_{k_3}(n_k + n_{k_1}) - n_k n_{k_1}(n_{k_2} + n_{k_3})\} \tag{6.3}$$

Substituting (6.3) and (6.2) into the equation for n_k we get the five-wave kinetic equation:

$$\frac{\partial n_k}{\partial t} = st(n,n,n) + st(n,n,n,n)$$

$$st(n,n,n) = \pi \int |T_{k_2 k_3}^{k k_1}|^2 f_{k_2 k_3}^{k k_1} dk_1 dk_2 dk_3$$

$$st(n,n,n,n) = \frac{\pi}{3} \int |T_{k k_5}^{k_1 k_2 k_3}|^2 f_{k k_5}^{k_1 k_2 k_3} dk_1 dk_2 dk_3 dk_5 - \frac{\pi}{2} \int |T_{k_4 k_5}^{k k_2 k_3}|^2 f_{k_4 k_5}^{k k_2 k_3} dk_2 dk_3 dk_4 dk_5,$$

$$f_{k_2 k_3}^{k k_1} = \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} \{n_{k_2} n_{k_3}(n_k + n_{k_1}) - n_k n_{k_1}(n_{k_2} + n_{k_3})\}$$

$$f_{k_4 k_5}^{k_1 k_2 k_3} = \delta_{k_1+k_2+k_3-k_4-k_5} \delta_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-\omega_{k_4}-\omega_{k_5}} \{n_{k_1} n_{k_2} n_{k_3}(n_{k_4} + n_{k_5}) - n_{k_4} n_{k_5}(n_{k_1} n_{k_2} + n_{k_1} n_{k_3} + n_{k_2} n_{k_3})\}$$

As it was shown in the Introduction, In the one dimensional case $st(n,n,n) \equiv 0$ and we end up with the pure five-wave kinetic equation:

$$\frac{\partial n_k}{\partial t} = st(n,n,n,n) \tag{6.4}$$

Eq. (6.4) formally preserves two integrals of motion, energy

$$E = \int_0^\infty \omega_k n_k dk,$$

and momentum

$$P = \int_0^\infty k n_k dk$$

(We consider the case when all k_i are positive.) The stationary equation

$$st(n,n,n,n) = 0 \tag{6.5}$$

has thermodynamic solution

$$n_k = \frac{T}{\omega_k + \alpha k}.$$

Like the four-wave isotropic kinetic equation, Eq. (6.4) describes direct and inverse cascades. The inverse cascade is the cascade of energy, which is a real constant of motion and is carried towards small k. It is described by the following Kolmogorov solution of Eq. (6.5)

$$n_k^{(1)} = \alpha^{(1)} \epsilon^{1/4} |k|^{-25/8}$$

Here ϵ is the energy flux, $\alpha^{(1)}$ is the Kolmogorov constant.

A corresponding energy spectrum is

$$\mathcal{E}_\omega d\omega = \omega_k n_k dk, \quad \mathcal{E}_\omega = \alpha^{(1)} \epsilon^{1/4} \omega^{-17/4}$$

Direct cascade is a transport of momentum towards the large wave numbers. It is described by the Kolmogorov solution

$$n_k^{(2)} = \alpha^{(2)}\mu^{1/4}|k|^{-13/4}$$

Here μ is the momentum flux, $\alpha^{(2)}$ is the Kolmogorov constant.

Now

$$\mathcal{E}_\omega = \alpha^{(2)}\mu^{1/4}\omega^{-9/2}$$

Due to the direct cascade the momentum is not a real constant of motion, it leaks permanently to the large k region. A more detailed description of the Kolmogorov spectra in the one-dimensional case will be published separately.

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Appendix A

A.1. 4th and 5th-order terms of the Hamiltonian

The first part of $\mathcal{H}_4$ is

$$\begin{aligned}\mathcal{H}_4{}^{1st} &= -\frac{1}{2}\int 2\Psi^{(3)}H\Psi^{(1)\prime}du = -\int \mathcal{P}(x'^2 - y'^2)H\mathcal{P}'du + 2\int x'H(y'\mathcal{P})H\mathcal{P}'du \\ &= \frac{1}{2\pi}\int |k_3|(|k_1k_2| + k_1k_2)z_{k_1}z_{k_2}\mathcal{P}_{k_3}\mathcal{P}_{k_4}\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4 \\ &\quad + \frac{1}{\pi}\int |k_1|k_2|k_3|s_{k_2+k_4}z_{k_1}z_{k_2}\mathcal{P}_{k_3}\mathcal{P}_{k_4}\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4\end{aligned}$$

The second part of $\mathcal{H}_4$ is

$$\mathcal{H}_4{}^{2nd} = -\frac{1}{2}\int \Psi^{(2)}H\Psi^{(2)\prime}du = -\frac{1}{2}\int (y'\mathcal{P})H(y'\mathcal{P})' + (x'\mathcal{P})H(x'\mathcal{P})' + 2(x'\mathcal{P})(y'\mathcal{P})'du$$

In the k-space it acquires the form:

$$\begin{aligned}\mathcal{H}_4{}^{2nd} &= \frac{1}{4\pi}\int \{(|k_1k_2| - k_1k_2)|k_2 + k_4| + (|k_1|k_2 - k_1|k_2|)(k_2 + k_4)\} \\ &\quad \times z_{k_1}z_{k_2}\mathcal{P}_{k_3}\mathcal{P}_{k_4}\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4 \qquad \text{(A.1.1)}\end{aligned}$$

The first part of $\mathcal{H}_5$ is

$$\begin{aligned}\mathcal{H}_5{}^{1st} &= -\int \Psi^{(4)}H\Psi^{(1)\prime}du = \frac{-1}{(2\pi)^{\frac{3}{2}}}\int |k_4|\{(3k_1k_2 + |k_1k_2|)|k_3| + (3|k_1k_2| + k_1k_2)s_{k_3+k_5}k_3\} \\ &\quad \times z_{k_1}z_{k_2}z_{k_3}\mathcal{P}_{k_4}\mathcal{P}_{k_5}\delta_{k_1+k_2+k_3+k_4+k_5}dk_1dk_2dk_3dk_4dk_5\end{aligned}$$

The second part of $\mathcal{H}_5$ is

$$\mathcal{H}_5^{\text{2nd}} = -\int \Psi^{(3)} H \Psi^{(2)\prime} du = \int \mathcal{P}(x'^2 - y'^2)(z'\mathcal{P})' du + \int \mathcal{P}(x'^2 - z'^2) H(x'\mathcal{P})' du$$
$$-2\int x' H(z'\mathcal{P})(z'\mathcal{P}')' du - 2\int x' H(z'\mathcal{P}) H(x'\mathcal{P})' du$$
$$= -\frac{1}{(2\pi)^{\frac{3}{2}}} \int \{(|k_1 k_2| + k_1 k_2)(k_3(k_3 + k_4) + |k_3||k_3 + k_4|)\}$$
$$\times z_{k_1} z_{k_2} z_{k_3} \mathcal{P}_{k_4} \mathcal{P}_{k_5} \delta_{k_1+k_2+k_3+k_4+k_5} dk_1 dk_2 dk_3 dk_4 dk_5$$
$$-\frac{2}{(2\pi)^{\frac{3}{2}}} \int |k_1| k_3 s_{k_3+k_4} (k_2(k_2 + k_5) + |k_2||k_2 + k_5|)$$
$$\times z_{k_1} z_{k_2} z_{k_3} \mathcal{P}_{k_4} \mathcal{P}_{k_5} \delta_{k_1+k_2+k_3+k_4+k_5} dk_1 dk_2 dk_3 dk_4 dk_5$$

The Hamiltonian can be written in symmetrical form:

$$\mathcal{H} = \frac{1}{2}\int (g|z_k|^2 + |k||\mathcal{P}_k|^2) dk + \frac{1}{2\sqrt{2\pi}} \int \{S_{k_1 k_2 k_3} z_{k_1} z_{k_2} z_{k_3} - F^{k_1}_{k_2 k_3} z_{k_1} \mathcal{P}_{k_2} \mathcal{P}_{k_3}\} \delta_{k_1+k_2+k_3} dk_1 dk_2 dk_3$$
$$+\frac{1}{4\pi} \int M^{k_1 k_2}_{k_3 k_4} z_{k_1} z_{k_2} \mathcal{P}_{k_3} \mathcal{P}_{k_4} \delta_{k_1+k_2+k_3+k_4} dk_1 dk_2 dk_3 dk_4$$
$$+\frac{1}{2(2\pi)^{\frac{3}{2}}} \int N^{k_1 k_2 k_3}_{k_4 k_5} z_{k_1} z_{k_2} z_{k_3} \mathcal{P}_{k_4} \mathcal{P}_{k_5} \delta_{k_1+k_2+k_3+k_4+k_5} dk_1 dk_2 dk_3 dk_4 dk_5 \qquad \text{(A.1.2)}$$

where $S_{k_1 k_2 k_3}$, $L_{k_1 k_2}$ and $F^{k_1}_{k_2 k_3}$ are defined in (3.2), and

$$M^{k_1 k_2}_{k_3 k_4} = (|k_3| + |k_4|)(k_1 k_2 + |k_1 k_2|) + \tfrac{1}{4} L_{-k_1 k_2}(|k_1 + k_3| + |k_1 + k_4| + |k_2 + k_3| + |k_2 + k_4|)$$
$$+\tfrac{1}{2}(|k_1| k_2(2k_2 + k_3 + k_4) + k_1|k_2|(2k_1 + k_3 + k_4))$$
$$+|k_1| k_2(|k_3| s_{k_2+k_4} + |k_4| s_{k_2+k_3}) + k_1|k_2|(|k_3| s_{k_1+k_4} + |k_4| s_{k_1+k_3})$$

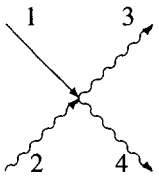
$$N^{k_1 k_2 k_3}_{k_4 k_5} = -\Big\{(|k_1 k_2 k_3| + |k_1| k_2 k_3 + k_1|k_2| k_3 + k_1 k_2|k_3|)(|k_4| + |k_5|)$$
$$+\frac{k_1 k_2 k_3}{3}(|k_4|(s_{k_1+k_5} + s_{k_2+k_5} + s_{k_3+k_5}) + |k_5|(s_{k_1+k_4} + s_{k_2+k_4} + s_{k_3+k_4}))$$
$$+|k_4|(k_1|k_2||k_3| s_{k_1+k_5} + |k_1| k_2|k_3| s_{k_2+k_5} + |k_1||k_2| k_3 s_{k_3+k_5})$$
$$+|k_5|(k_1|k_2||k_3| s_{k_1+k_4} + |k_1| k_2|k_3| s_{k_2+k_4} + |k_1||k_2| k_3 s_{k_3+k_4})\Big\}$$

$$-\tfrac{1}{3}\{L_{k_1,k_2} L_{k_3,k_3+k_4} + L_{k_3,k_2} L_{k_1,k_1+k_4} + L_{k_1,k_3} L_{k_2,k_2+k_4}$$
$$+L_{k_1,k_2} L_{k_3,k_3+k_5} + L_{k_3,k_2} L_{k_1,k_1+k_5} + L_{k_1,k_3} L_{k_2,k_2+k_5}\}$$
$$-\tfrac{1}{3}\{|k_1|(k_3(L_{k_2,k_2+k_4} s_{k_3+k_5} + L_{k_2,k_2+k_5} s_{k_3+k_4}) + k_2(L_{k_3,k_3+k_4} s_{k_2+k_5} + L_{k_3,k_3+k_5} s_{k_2+k_4}))$$
$$+|k_2|(k_3(L_{k_1,k_1+k_4} s_{k_3+k_5} + L_{k_1,k_1+k_5} s_{k_3+k_4}) + k_1(L_{k_3,k_3+k_4} s_{k_1+k_5} + L_{k_3,k_3+k_5} s_{k_1+k_4}))$$
$$+|k_3|(k_2(L_{k_1,k_1+k_4} s_{k_2+k_5} + L_{k_1,k_1+k_5} s_{k_2+k_4}) + k_1(L_{k_2,k_2+k_4} s_{k_1+k_5} + L_{k_2,k_2+k_5} s_{k_1+k_4}))\}$$

Appendix B

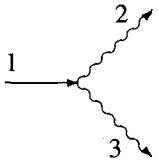
Let us introduce basic object of diagrammatic technique we use in this work.

(i) Bare fourth order vertex with 2 incoming and 2 outgoing wave vectors:

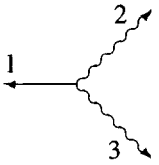


$$W_{k_3 k_4}^{k_1 k_2}$$

(ii) Bare third order vertexes U and V:

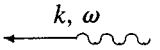


$$V_{k_2 k_3}^{k_1}$$



$$U_{k_1 k_2 k_3}$$

(iii) Bare Green function $G(k, \omega)$.

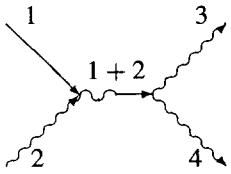


$$G(k, \omega) = \frac{1}{\omega - \omega_k}$$

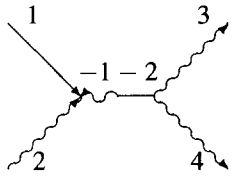
Note, that each vertex has just one straight line and others are wavy.

In order to calculate fourth order interaction matrix element we have to add to bare fourth order vertex all possible combinations of lower order vertexes (third order in this particular case) connected with Green function in such a way, that the resulting diagrams have 2 incoming and 2 outgoing wave vectors and having no internal loops.

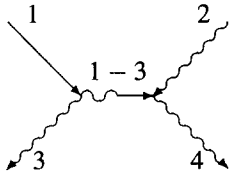
It is easy to see, that the only way to fulfill these requirements is to connect 2 third order vertexes by one Green function. As the result we have 6 topologically different arrangements. The arguments k and ω of internal Green function should be calculated from resonant conditions (1.1). Since we are on the resonant manifold it does not matter do we calculate arguments k and ω of Green function from left or from right vertex, because it they both give the same result. This reflects the fact that two ratios in each line in square brackets of (4.3) are equal to each others. This removes extra $1/2$.



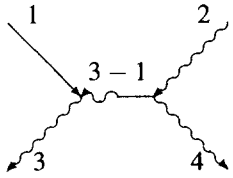
$$-V_{k_1 k_2}^{k_1+k_2} V_{k_3 k_4}^{k_3+k_4} \left[\frac{1}{\omega_{k_1+k_2} - \omega_{k_1} - \omega_{k_2}} \right]$$



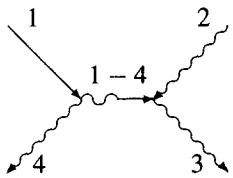
$$-U_{-k_1-k_2k_1k_2}U_{-k_3-k_4k_3k_4}\left[\frac{1}{\omega_{k_1+k_2}+\omega_{k_1}+\omega_{k_2}}\right]$$



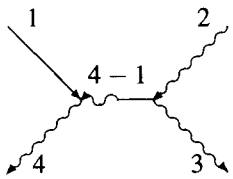
$$-V^{k_1}_{k_3k_1-k_3}V^{k_4}_{k_2k_4-k_2}\left[\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}}\right]$$



$$-V^{k_2}_{k_4k_2-k_4}V^{k_3}_{k_1k_3-k_1}\left[\frac{1}{\omega_{k_4}+\omega_{k_2-k_4}-\omega_{k_2}}\right]$$



$$-V^{k_1}_{k_4k_1-k_4}V^{k_3}_{k_2k_3-k_2}\left[\frac{1}{\omega_{k_4}+\omega_{k_1-k_4}-\omega_{k_1}}\right]$$

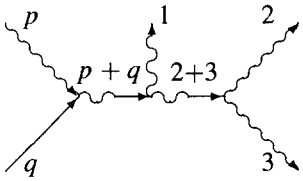


$$-V^{k_2}_{k_3k_2-k_3}V^{k_4}_{k_1k_4-k_1}\left[\frac{1}{\omega_{k_3}+\omega_{k_2-k_3}-\omega_{k_2}}\right]$$

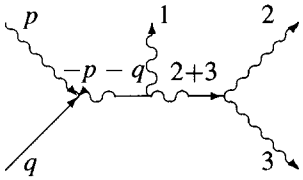
Appendix C

Following the same steps as in the Appendix B, we can construct diagrammatic series for five order matrix element $T^{kk_1k_2}_{k_3k_4}$. We have to combine all third and fourth order vertexes in such a way that we have 2 "incoming" arguments and 3 "out-coming" arguments. The result of course will be the same if we would consider 3 "incoming" arguments and 2 "out-coming" arguments. Considering all possible topologies consistent with definitions of vertexes and Green functions and without internal loops, we conclude, that there exist 60 diagrams constructed from three third order vertices and 2 Green functions and 20 diagrams constructed from one three order vertex, one four order vertex and one Green function. We call these two groups "3+3+3" and "4+3" correspondingly.

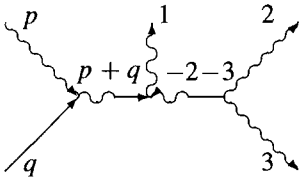
Below are the diagrams and analytical expressions for "3+3+3" and "4+3" terms. Together with the bare fifth order vertex $Q^{k_1k_2k_3}_{pq}$ this sum gives the full fifth-order interaction matrix element $T^{k_1k_2k_3}_{pq}$ or $T^{pq}_{k_1k_2k_3}$. We used these expressions as an input to Matematica, therefore notation here is slightly different.



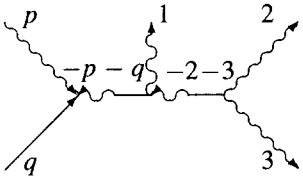
$$\frac{V(k2+k3,k2,k3)\,V(p+q,k1,k2+k3)\,V(p+q,p,q)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\;(\omega(p)+\omega(q)-\omega(p+q))}$$



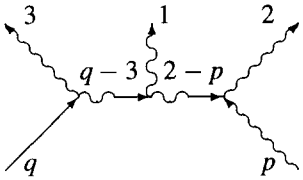
$$-\frac{U(k1,-p-q,k2+k3)\,U(p,q,-p-q)\,V(k2+k3,k2,k3)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\;(\omega(p)+\omega(q)+\omega(p+q))}$$



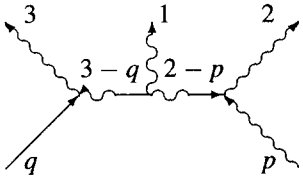
$$-\frac{U(k2,k3,-k2-k3)\,V(k1,p+q,-k2-k3)\,V(p+q,p,q)}{(\omega(k2)+\omega(k3)+\omega(k2+k3))\;(\omega(p)+\omega(q)-\omega(p+q))}$$



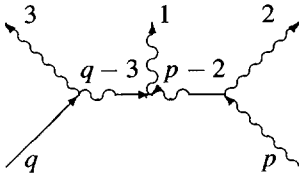
$$\frac{U(k2,k3,-k2-k3)\,U(p,q,-p-q)\,V(-k2-k3,k1,-p-q)}{(\omega(k2)+\omega(k3)+\omega(k2+k3))\;(\omega(p)+\omega(q)+\omega(p+q))}$$



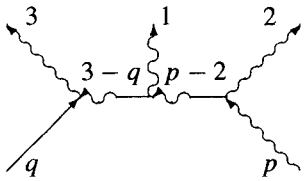
$$\frac{V(k2,p,k2-p)\,V(q,k3,-k3+q)\,V(-k3+q,k1,k2-p)}{(\omega(k2)-\omega(k2-p)-\omega(p))\;(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



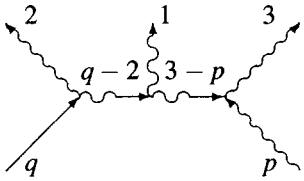
$$\frac{U(k1,k3-q,k2-p)\,V(k2,p,k2-p)\,V(k3,q,k3-q)}{(\omega(k2)-\omega(k2-p)-\omega(p))\;(\omega(k3)-\omega(k3-q)-\omega(q))}$$



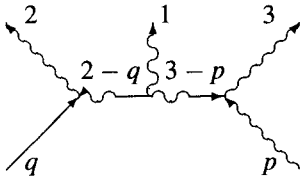
$$\frac{V(k1,-k3+q,-k2+p)\,V(p,k2,-k2+p)\,V(q,k3,-k3+q)}{(-\omega(k2)-\omega(k2-p)+\omega(p))\;(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



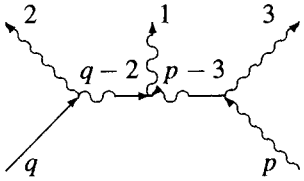
$$\frac{V(k3,q,k3-q)\,V(p,k2,-k2+p)\,V(-k2+p,k1,k3-q)}{(-\omega(k2)-\omega(k2-p)+\omega(p))\;(\omega(k3)-\omega(k3-q)-\omega(q))}$$



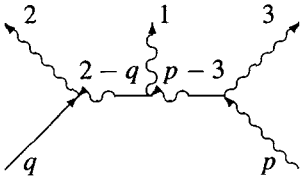
$$\frac{V(k3,p,k3-p)\,V(q,k2,-k2+q)\,V(-k2+q,k1,k3-p)}{(\omega(k3)-\omega(k3-p)-\omega(p))\;(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



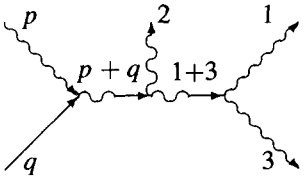
$$\frac{U(k1,k2-q,k3-p)\,V(k2,q,k2-q)\,V(k3,p,k3-p)}{(\omega(k3)-\omega(k3-p)-\omega(p))\;(\omega(k2)-\omega(k2-q)-\omega(q))}$$



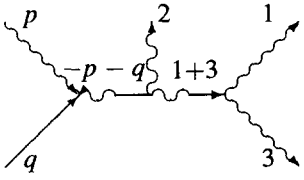
$$\frac{V(k1,-k2+q,-k3+p)\,V(p,k3,-k3+p)\,V(q,k2,-k2+q)}{(-\omega(k3)-\omega(k3-p)+\omega(p))\;(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



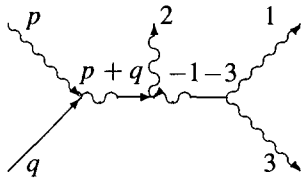
$$\frac{V(k2,q,k2-q)\,V(p,k3,-k3+p)\,V(-k3+p,k1,k2-q)}{(-\omega(k3)-\omega(k3-p)+\omega(p))\;(\omega(k2)-\omega(k2-q)-\omega(q))}$$



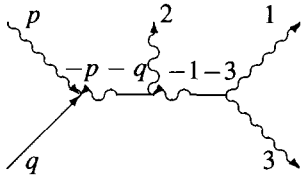
$$\frac{V(k1+k3,k1,k3)\,V(p+q,k2,k1+k3)\,V(p+q,p,q)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(\omega(p)+\omega(q)-\omega(p+q))}$$



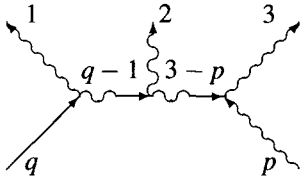
$$\frac{U(k2,k1+k3,-p-q)\,U(p,q,-p-q)\,V(k1+k3,k1,k3)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(-\omega(p)-\omega(q)-\omega(p+q))}$$



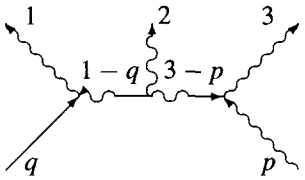
$$\frac{U(k1,k3,-k1-k3)\,V(k2,p+q,-k1-k3)\,V(p+q,p,q)}{(-\omega(k1)-\omega(k3)-\omega(k1+k3))\,(\omega(p)+\omega(q)-\omega(p+q))}$$



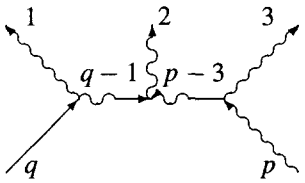
$$\frac{U(-k1-k3,k1,k3)\,U(p,q,-p-q)\,V(-k1-k3,k2,-p-q)}{(-\omega(k1)-\omega(k3)-\omega(k1+k3))\,(-\omega(p)-\omega(q)-\omega(p+q))}$$



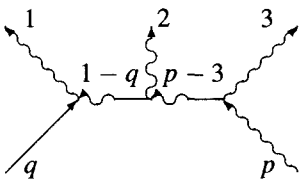
$$\frac{V(k3,p,k3-p)\,V(q,k1,-k1+q)\,V(-k1+q,k2,k3-p)}{(\omega(k3)-\omega(k3-p)-\omega(p))\,(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



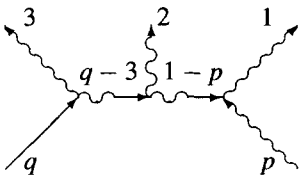
$$\frac{U(k2,k1-q,k3-p)\,V(k1,q,k1-q)\,V(k3,p,k3-p)}{(\omega(k3)-\omega(k3-p)-\omega(p))\,(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



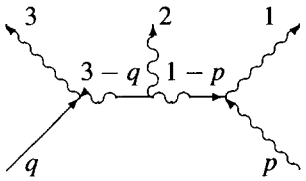
$$\frac{V(k2,-k1+q,-k3+p)\,V(p,k3,-k3+p)\,V(q,k1,-k1+q)}{(-\omega(k3)-\omega(k3-p)+\omega(p))\,(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



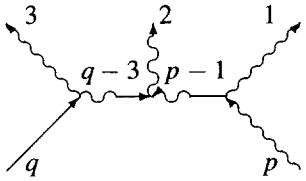
$$\frac{V(k1,q,k1-q)\,V(p,k3,-k3+p)\,V(-k3+p,k2,k1-q)}{(-\omega(k3)-\omega(k3-p)+\omega(p))\,(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



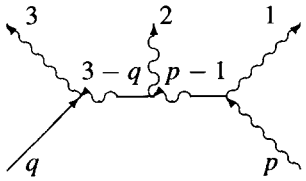
$$\frac{V(k1,p,k1-p)\,V(q,k3,-k3+q)\,V(-k3+q,k2,k1-p)}{(\omega(k1)-\omega(k1-p)-\omega(p))\,(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



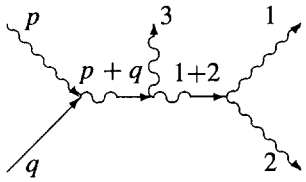
$$\frac{U(k2,k3-q,k1-p)\,V(k1,p,k1-p)\,V(k3,k3-q,q)}{(\omega(k1)-\omega(k1-p)-\omega(p))\,(\omega(k3)-\omega(k3-q)-\omega(q))}$$



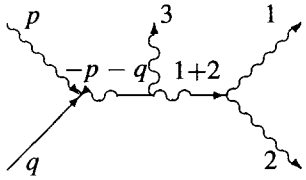
$$\frac{V(k2,-k3+q,-k1+p)\,V(p,k1,-k1+p)\,V(q,k3,-k3+q)}{(-\omega(k1)-\omega(k1-p)+\omega(p))\,(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



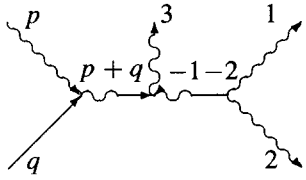
$$\frac{V(k3,k3-q,q)\,V(p,k1,-k1+p)\,V(-k1+p,k3-q,k2)}{(-\omega(k1)-\omega(k1-p)+\omega(p))\,(\omega(k3)-\omega(k3-q)-\omega(q))}$$



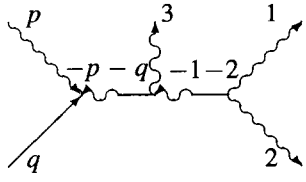
$$\frac{V(k1+k2,k1,k2)\,V(p+q,k3,k1+k2)\,V(p+q,p,q)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(\omega(p)+\omega(q)-\omega(p+q))}$$



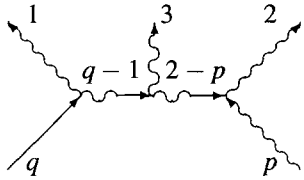
$$\frac{U(k3,k1+k2,-p-q)\,U(p,q,-p-q)\,V(k1+k2,k1,k2)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(-\omega(p)-\omega(q)-\omega(p+q))}$$



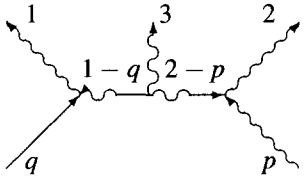
$$\frac{U(k1,k2,-k1-k2)\,V(k3,p+q,-k1-k2)\,V(p+q,p,q)}{(-\omega(k1)-\omega(k2)-\omega(k1+k2))\,(\omega(p)+\omega(q)-\omega(p+q))}$$



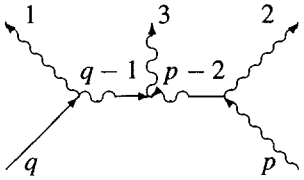
$$\frac{U(k1,k2,-k1-k2)\,U(p,q,-p-q)\,V(-k1-k2,k3,-p-q)}{(-\omega(k1)-\omega(k2)-\omega(k1+k2))\,(-\omega(p)-\omega(q)-\omega(p+q))}$$



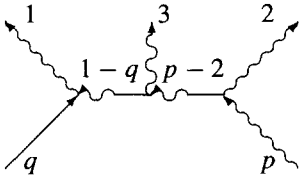
$$\frac{V(k2,p,k2-p)\,V(q,k1,-k1+q)\,V(-k1+q,k3,k2-p)}{(\omega(k2)-\omega(k2-p)-\omega(p))\,(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



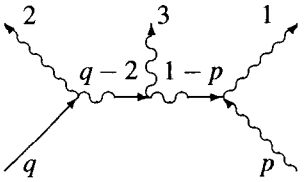
$$\frac{U(k1-q,k3,k2-p)\,V(k1,q,k1-q)\,V(k2,p,k2-p)}{(\omega(k2)-\omega(k2-p)-\omega(p))\,(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



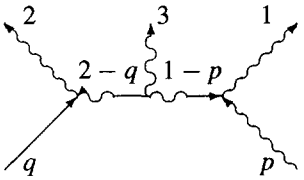
$$\frac{V(k3,-k1+q,-k2+p)\,V(p,k2,-k2+p)\,V(q,k1,-k1+q)}{(-\omega(k2)-\omega(k2-p)+\omega(p))\,(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



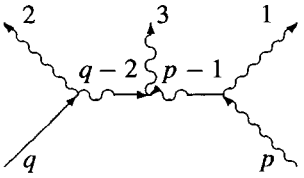
$$\frac{V(k1,q,k1-q)\,V(p,k2,-k2+p)\,V(-k2+p,k3,k1-q)}{(-\omega(k2)-\omega(k2-p)+\omega(p))\,(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



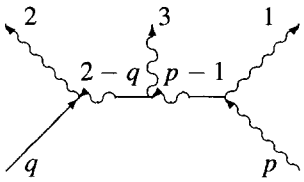
$$\frac{V(k1,p,k1-p)\,V(q,k2,-k2+q)\,V(-k2+q,k3,k1-p)}{(\omega(k1)-\omega(k1-p)-\omega(p))\,(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



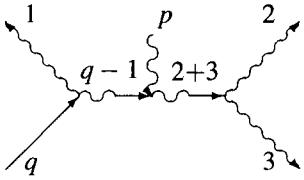
$$\frac{U(k3,k2-q,k1-p)\,V(k1,p,k1-p)\,V(k2,q,k2-q)}{(\omega(k1)-\omega(k1-p)-\omega(p))\,(\omega(k2)-\omega(k2-q)-\omega(q))}$$



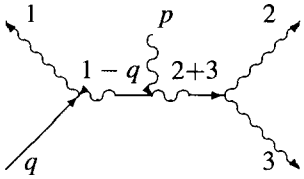
$$\frac{V(k3,-k2+q,-k1+p)\,V(p,k1,-k1+p)\,V(q,k2,-k2+q)}{(-\omega(k1)-\omega(k1-p)+\omega(p))\,(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



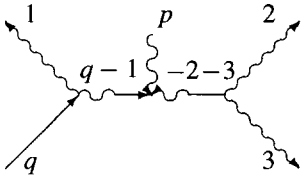
$$\frac{V(k2,q,k2-q)\,V(p,k1,-k1+p)\,V(-k1+p,k3,k2-q)}{(-\omega(k1)-\omega(k1-p)+\omega(p))\;(\omega(k2)-\omega(k2-q)-\omega(q))}$$



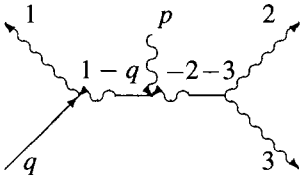
$$\frac{V(k2+k3,k2,k3)\,V(k2+k3,p,-k1+q)\,V(q,k1,-k1+q)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\;(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



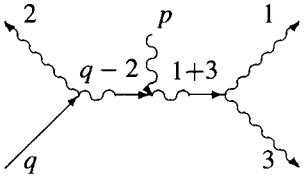
$$\frac{V(k1,q,k1-q)\,V(k2+k3,k2,k3)\,V(p,k1-q,k2+k3)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\;(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



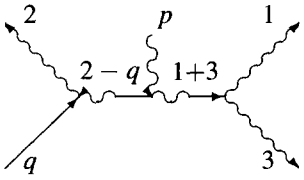
$$\frac{U(k2,k3,-k2-k3)\,U(p,-k1+q,-k2-k3)\,V(q,k1,-k1+q)}{(-\omega(k2)-\omega(k3)-\omega(k2+k3))\;(-\omega(k1)+\omega(q)-\omega(-k1+q))}$$



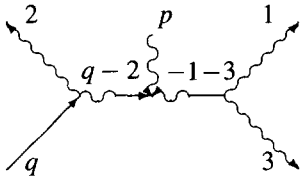
$$\frac{U(k2,k3,-k2-k3)\,V(k1,q,k1-q)\,V(k1-q,p,-k2-k3)}{(-\omega(k2)-\omega(k3)-\omega(k2+k3))\;(\omega(k1)-\omega(q)-\omega(-k1+q))}$$



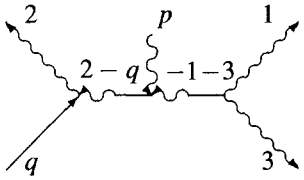
$$\frac{V(k1+k3,k1,k3)\,V(k1+k3,p,-k2+q)\,V(q,k2,-k2+q)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



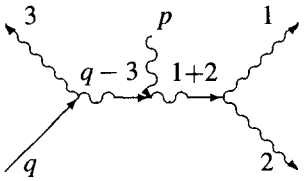
$$\frac{V(k2,q,k2-q)\,V(k1+k3,k1,k3)\,V(p,k2-q,k1+k3)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(\omega(k2)-\omega(k2-q)-\omega(q))}$$



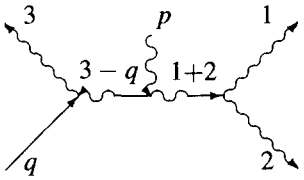
$$\frac{U(k1,k3,-k1-k3)\,U(p,-k2+q,-k1-k3)\,V(q,k2,-k2+q)}{(-\omega(k1)-\omega(k3)-\omega(k1+k3))\,(-\omega(k2)-\omega(k2-q)+\omega(q))}$$



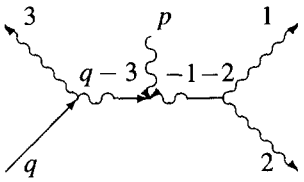
$$-\frac{U(k1,k3,-k1-k3)\,V(k2,q,k2-q)\,V(k2-q,p,-k1-k3)}{(\omega(k1)+\omega(k3)+\omega(k1+k3))\,(\omega(k2)-\omega(k2-q)-\omega(q))}$$



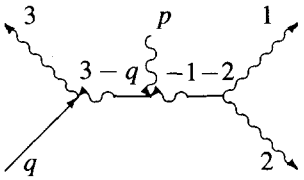
$$\frac{V(k1+k2,k1,k2)\,V(k1+k2,p,-k3+q)\,V(q,k3,-k3+q)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



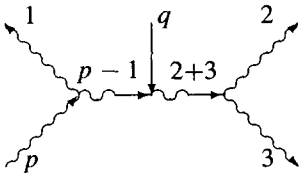
$$\frac{V(k1+k2,k1,k2)\,V(k3,q,k3-q)\,V(p,k3-q,k1+k2)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(\omega(k3)-\omega(k3-q)-\omega(q))}$$



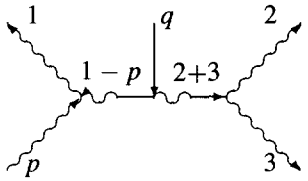
$$\frac{U(-k1-k2,k1,k2)\,U(-k3+q,-k1-k2,p)\,V(q,k3,-k3+q)}{(-\omega(k1)-\omega(k2)-\omega(k1+k2))\,(-\omega(k3)-\omega(k3-q)+\omega(q))}$$



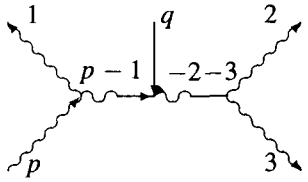
$$-\frac{U(k1,k2,-k1-k2)\,V(k3,q,k3-q)\,V(k3-q,p,-k1-k2)}{(\omega(k1)+\omega(k2)+\omega(k1+k2))\,(\omega(k3)-\omega(k3-q)-\omega(q))}$$



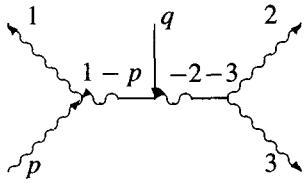
$$\frac{V(k2+k3,k2,k3)\,V(k2+k3,q,-k1+p)\,V(p,k1,-k1+p)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\,(-\omega(k1)-\omega(k1-p)+\omega(p))}$$



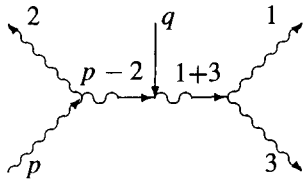
$$\frac{V(k1,p,k1-p)\,V(k2+k3,k2,k3)\,V(q,k1-p,k2+k3)}{(\omega(k2)+\omega(k3)-\omega(k2+k3))\;(\omega(k1)-\omega(k1-p)-\omega(p))}$$



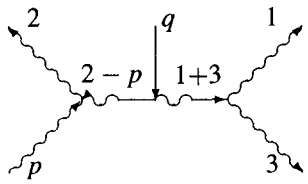
$$\frac{U(k2,k3,-k2-k3)\,U(q,-k1+p,-k2-k3)\,V(p,k1,-k1+p)}{(-\omega(k2)-\omega(k3)-\omega(k2+k3))\;(-\omega(k1)-\omega(k1-p)+\omega(p))}$$



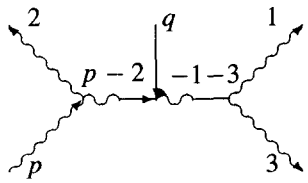
$$\frac{U(-k2-k3,k2,k3)\,V(k1,p,k1-p)\,V(k1-p,q,-k2-k3)}{(-\omega(k2)-\omega(k3)-\omega(k2+k3))\;(\omega(k1)-\omega(k1-p)-\omega(p))}$$



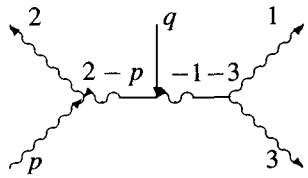
$$\frac{V(k1+k3,k1,k3)\,V(k1+k3,q,-k2+p)\,V(p,k2,-k2+p)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(-\omega(k2)-\omega(k2-p)+\omega(p))}$$



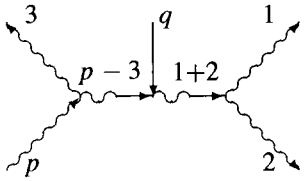
$$\frac{V(k2,p,k2-p)\,V(k1+k3,k1,k3)\,V(q,k2-p,k1+k3)}{(\omega(k1)+\omega(k3)-\omega(k1+k3))\;(\omega(k2)-\omega(k2-p)-\omega(p))}$$



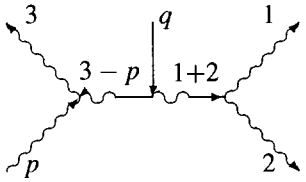
$$\frac{U(-k1-k3,k1,k3)\,U(-k2+p,q,-k1-k3)\,V(p,k2,-k2+p)}{(-\omega(k1)-\omega(k3)-\omega(k1+k3))\;(-\omega(k2)-\omega(k2-p)+\omega(p))}$$



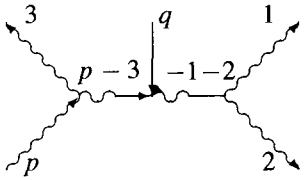
$$\frac{U(k1,k3,-k1-k3)\,V(k2,p,k2-p)\,V(k2-p,q,-k1-k3)}{(-\omega(k1)-\omega(k3)-\omega(k1+k3))\;(\omega(k2)-\omega(k2-p)-\omega(p))}$$



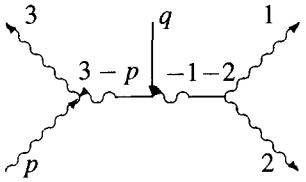
$$\frac{V(k1+k2,k1,k2)\,V(k1+k2,q,-k3+p)\,V(p,k3,-k3+p)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(-\omega(k3)-\omega(k3-p)+\omega(p))}$$



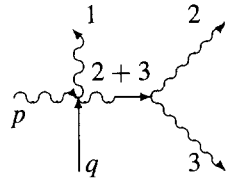
$$\frac{V(k3,p,k3-p)\,V(q,k3-p,k1+k2)\,V(k1+k2,k1,k2)}{(\omega(k1)+\omega(k2)-\omega(k1+k2))\,(\omega(k3)-\omega(k3-p)-\omega(p))}$$



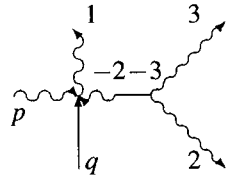
$$\frac{U(k1,k2,-k1-k2)\,U(q,-k3+p,-k1-k2)\,V(p,k3,-k3+p)}{(-\omega(k1)-\omega(k2)-\omega(k1+k2))\,(-\omega(k3)-\omega(k3-p)+\omega(p))}$$



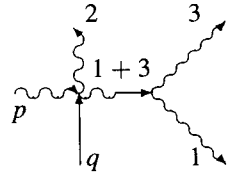
$$\frac{V(k3,p,k3-p)\,V(k3-p,q,-k1-k2)\,U(k1,k2,-k1-k2)}{(-\omega(k1)-\omega(k2)-\omega(k1+k2))\,(\omega(k3)-\omega(k3-p)-\omega(p))}$$



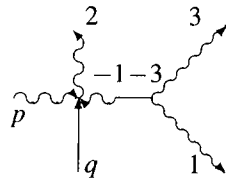
$$\frac{V(k2+k3,k2,k3)\,W(p,q,k1,k2+k3)}{\omega(k2)+\omega(k3)-\omega(k2+k3)}$$



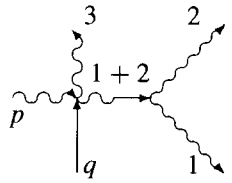
$$-\frac{G(k1,p,q,-k2-k3)\,U(k2,k3,-k2-k3)}{\omega(k2)+\omega(k3)+\omega(k2+k3)}$$



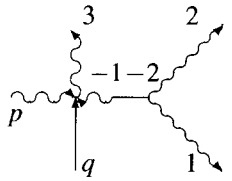
$$\frac{V(k1+k3,k1,k3)\,W(p,q,k2,k1+k3)}{\omega(k1)+\omega(k3)-\omega(k1+k3)}$$



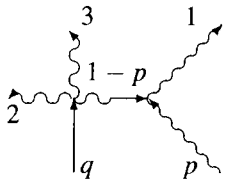
$$-\frac{G(k2,p,q,-k1-k3)\,U(k1,k3,-k1-k3)}{\omega(k1)+\omega(k3)+\omega(k1+k3)}$$



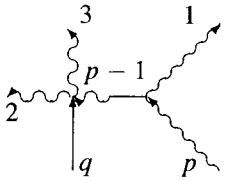
$$\frac{V(k1+k2,k1,k2)\,W(p,q,k3,k1+k2)}{\omega(k1)+\omega(k2)-\omega(k1+k2)}$$



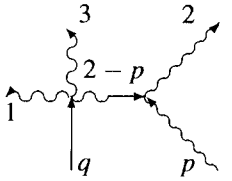
$$-\frac{G(k3,p,q,-k1-k2)\,U(-k1-k2,k1,k2)}{\omega(k1)+\omega(k2)+\omega(k1+k2)}$$



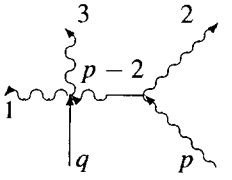
$$\frac{G(q,k1-p,k2,k3)\,V(k1,p,k1-p)}{\omega(k1)-\omega(k1-p)-\omega(p)}$$



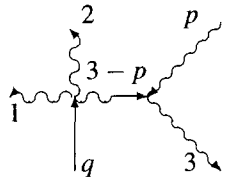
$$\frac{V(p,k1,-k1+p)\,W(q,-k1+p,k2,k3)}{-\omega(k1)-\omega(k1-p)+\omega(p)}$$



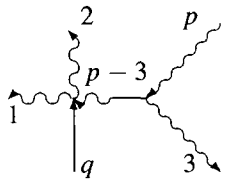
$$\frac{G(q,k2-p,k1,k3)\,V(k2,p,k2-p)}{\omega(k2)-\omega(k2-p)-\omega(p)}$$



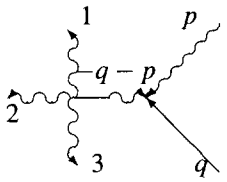
$$\frac{V(p,k2,-k2+p)\,W(q,-k2+p,k1,k3)}{-\omega(k2)-\omega(k2-p)+\omega(p)}$$



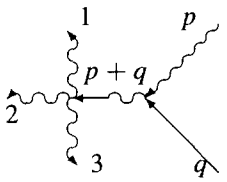
$$\frac{G(q,k3-p,k1,k2)\,V(k3,p,k3-p)}{\omega(k3)-\omega(k3-p)-\omega(p)}$$



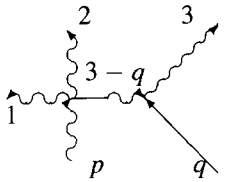
$$\frac{V(p,k3,-k3+p)\,W(q,-k3+p,k2,k1)}{-\omega(k3)-\omega(k3-p)+\omega(p)}$$



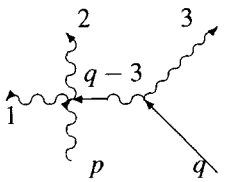
$$-\frac{R(k1,k2,k3,-p-q)\,U(-p-q,p,q)}{\omega(p)+\omega(q)+\omega(p+q)}$$



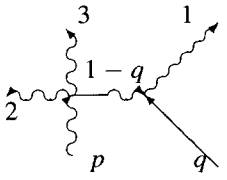
$$\frac{G(p+q,k1,k2,k3)\,V(p+q,p,q)}{\omega(p)+\omega(q)-\omega(p+q)}$$



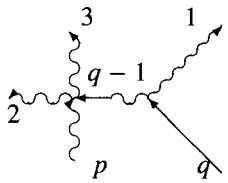
$$\frac{G(p,k1,k2,k3-q)\,V(k3,q,k3-q)}{\omega(k3)-\omega(k3-q)-\omega(q)}$$



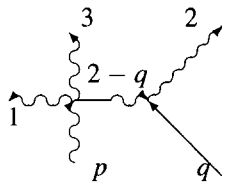
$$\frac{V(q,k3,-k3+q)\,W(p,-k3+q,k1,k2)}{-\omega(k3)-\omega(k3-q)+\omega(q)}$$



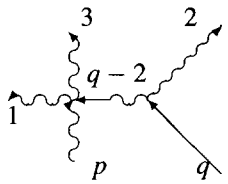
$$\frac{G(p,k2,k3,k1-q)\,V(k1,q,k1-q)}{\omega(k1)-\omega(q)-\omega(-k1+q)}$$



$$\frac{V(q,k1,-k1+q)\,W(p,-k1+q,k2,k3)}{-\omega(k1)+\omega(q)-\omega(-k1+q)}$$



$$\frac{G(p,k1,k3,k2-q)\,V(k2,q,k2-q)}{\omega(k2)-\omega(k2-q)-\omega(q)}$$



$$\frac{V(q,k2,-k2+q)\,W(p,-k2+q,k1,k3)}{-\omega(k2)-\omega(k2-q)+\omega(q)}$$

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