

Research paper

Exact discrete resonances in the Fermi-Pasta-Ulam–Tsingou system

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ABSTRACT

In systems of N coupled anharmonic oscillators, exact resonant interactions play an important role in the exchange of energy between normal modes. In the weakly nonlinear regime, those interactions may be responsible for the equipartition of energy in Fourier space. Here we consider analytically resonant wave-wave interactions for the celebrated Fermi-Pasta-Ulam-Tsingou (FPUT) system. Using a number-theoretical approach based on *cyclotomic polynomials*, we show that the problem of finding exact resonances for a system of N particles is equivalent to a Diophantine equation whose solutions depend sensitively on the number of particles N , in particular on the set of divisors of N . We provide an algorithm to construct *all* possible resonances for N particles, based on two basic methods: pairing-off and cyclotomic, which we introduce and use to build up explicit solutions to the 4-wave, 5-wave and 6-wave resonant conditions. Our results shed some light in the understanding of the long-standing FPUT paradox, regarding the sensitivity of the resonant manifolds with respect to the number of particles N and the corresponding time scale of the interactions leading to an eventual thermalisation. In this light we demonstrate that 6-wave resonances always exist for any N , while 5-wave resonances exist if N is divisible by 3 and $N \geq 9$. It is known that in the discrete case 4-wave resonances do not produce energy mixing across the spectrum, so we investigate whether 5-wave resonances can produce energy mixing across a significant region of the Fourier spectrum by looking at the structure of the interconnected network of Fourier modes that can interact nonlinearly via resonances. We obtain that the answer depends on the set of odd divisors of N which are not divisible by 3: the size of this set determines the number of dynamically independent components, corresponding to independent constants of motion (energies). We show that 6-wave resonances connect all these independent components, providing in principle a restoring mechanism for full-scale thermalisation.

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1. Introduction

During the early fifties, Fermi, Pasta, Ulam and Tsingou were interested in understanding the role of a weak nonlinearity in the phenomenon of thermalisation and transport mechanisms exhibited by real solids [1]. They considered a simplified

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model consisting of N identical masses connected by a nonlinear spring; the elastic force between the masses is expressed as a power series in the spring deformation Δx :

$$F = -\kappa \Delta x + \alpha (\Delta x)^2 + \beta (\Delta x)^3 + \dots, \quad (1)$$

where κ , α and β are elastic, spring dependent, constants. One can recognise that for small displacements the classical Hooke's law is recovered. They performed a number of numerical simulations with initial conditions characterised by energy concentrated on a single mode. They were expecting that, after some time, the nonlinearity would have played the key role of re-distributing the energy among different modes up to its equipartition. The results they obtained were somehow disappointing because, after some initial sharing of energy between a few modes, energy returned to the initial condition. The recurrence time observed has been named *FPU recurrence*, after the work of Fermi et al. [1] (recently it has been recognised that also Mary Tsingou took part to the research and sometimes the recurrence is named *FPUT* [2]). This highly remarkable result underlines the fact that nonlinearity is not sufficient for guaranteeing the phenomenon of equipartition. The results [1] were followed by a number of developments: in the sixties Zabusky and Kruskal studied the long wave limit of the FPUT chain and, using numerical simulations, they made the exceptional discovery of solitons [3], which was followed by the discovery of integrable partial differential equations. Those systems by definition do not reach a thermalisation state because the motion in phase space is confined on invariant tori. The knowledge of integrable systems did help only marginally to the understanding of the observed recurrent phenomena in the original FPUT system, the latter being a non integrable system.

Another attempt to explain the numerical results was based on the KAM theorem, presented by Kolmogorov more or less in the same years as the FPUT studies. Loosely speaking, under some specific assumptions, the theorem states that a slightly perturbed integrable Hamiltonian remain quasi-periodic. The KAM theorem appeared a natural tool to estimate the time scale of equipartition; however, to our knowledge, no quantitative prediction on the time scale has been made. Its applicability to the FPUT system has been established only recently by [4]. More precisely, it has been shown that the β -Fermi-Pasta-Ulam-Tsingou lattice with fixed endpoints and an arbitrary finite number of moving particles has a completely integrable Birkhoff normal form, which is an integrable approximation to the original Hamiltonian function. The result was extended to the $(\alpha + \beta)$ -FPUT system in [5].

In 1966 a remarkable result was obtained by Izrailev and Chirikov [6] who developed the concept of overlap of nonlinear resonances. The idea behind it is that for small nonlinearity, the perturbation to the unperturbed Hamiltonian is small and motion should be of quasi-periodic nature, as observed in the numerics by Fermi, Pasta, Ulam and Tsingou. However, by increasing the amplitude of the perturbation, one expects that stochasticity appears. Izrailev and Chirikov were able to estimate the limit of stochasticity. This theoretical prediction was successfully tested numerically in different papers (see for example [7] and more recent developments in [8]). In [9], the authors studied numerically the β -FPUT system considering as initial condition the mode corresponding to the highest frequency and found that above an analytically derived energy threshold, such mode is shown to be modulationally unstable; the instability gives rise to the formation of chaotic breathers whose life time is related to the time for the system to reach equipartition.

All the above scenarios rely on the existence of an energy threshold above which a thermalisation is observed. Recent numerical simulations (see for example [10,11]) have shown that, even for very small nonlinearity, equipartition is reached. The evolution of the spectrum is characterised by two stages: a first one in which the spectrum assumes an exponential decay (this is called metastable state, after [12]) and the second one characterised by the observation of thermalisation. Recently, in [13] and [8] the wave turbulence approach to the α -FPUT and β -FPUT system was considered and the problem with $N=16, 32$ and 64 masses was addressed. The main idea under such approach is that, in the weakly nonlinear limit, exact resonant interactions are responsible for an irreversible transfer of energy which leads to equipartition (similar results have been obtained for the discrete nonlinear Klein-Gordon equation [14]). By exact resonant interactions between M waves we mean that a wave scattering process characterised by the following conservation laws (momentum, k , and energy, $\omega(k)$)

$$\begin{aligned} k_1 \pm k_2 \pm \dots \pm k_M &= 0 \\ \omega(k_1) \pm \omega(k_2) \pm \dots \pm \omega(k_M) &= 0 \end{aligned} \quad (2)$$

is possible. For the specific case of a discrete lattice, wave numbers k_i are integer values and the equality in the first equation should be considered in modulo N . In [8,13] it was shown that in the β - and α -FPUT systems with $N=32$ the dominant interaction is a six-wave one, i.e. $M=6$. Based on the kinetic description of waves [15], a nonlinear time scale is associated to each resonant process; consequently, it is possible to estimate the time scale associated with the equipartition. In [13] it has been also conjectured that the dominant interaction problem is strongly dependent on the number of masses, N , of the chain. Therefore, finding the value of M for a given number of particles N is of fundamental importance for understanding the relaxation dynamics. However, establishing the lowest value of M at which the system of Eq. (2) has solutions is not an easy task. In [13] the existence of the solutions and the solution themselves were found numerically; the numerical approach is feasible when the number of particles is small and soon becomes prohibitive as the number of particles increases.

In this paper we attack the problem from an analytical point of view and show that the problem of finding M -wave resonances reduces to the solution of a set of Diophantine equations for which we develop an algorithm for finding the solutions. The paper is organised as follows: Section 2 is devoted to an introduction to the $(\alpha + \beta)$ -FPUT physical model in the so called *wave action* variables; we also give an overview of the wave-wave interaction theory in the weakly-nonlinear limit. In

Section 3, the general problem of finding solutions to the resonance conditions (2) is presented and discussed: basic definitions of irreducible and composite resonances, elementary results such as forbidden processes (3-wave resonances as well as processes converting $M - 1$ to 1 waves, for any $M \geq 3$), and a preliminary discussion advocating that in the weakly-nonlinear limit it is necessary and sufficient to search for M -wave resonances for $4 \leq M \leq 6$. In **Section 4** we introduce the *pairing-off method*, which provides the simplest type of solutions to the resonance conditions (2), valid for 2S-wave resonances. This method produces the well-known 4-wave resonances, and generalises the recently found 6-wave resonances for $N = 2^b$ to the case of N arbitrary. The pairing-off method turns out to be an essential part of the more complex methods discussed in the rest of the paper. In **Section 5** we show, with a simple example of 6-wave interactions in the case when the number of particles is $N = 6$, that not all solutions that exist are in pairing-off form, which serves to motivate our introduction of the *cyclotomic method* in **Sections 6** and **7**. The main result of **Section 6** is to simplify enormously the resonance conditions (2) in terms of polynomials in one complex variable $P(x)$ with integer coefficients, which evaluate to real values when x is equal to the primitive $2N$ -th root of unity $\zeta = \exp(i\pi/N)$, where N is the number of particles. The resonance conditions reduce to a simple linear Diophantine equation for a set of unknown integers, of easy solution. **Sections 7** and **8** provide for the first time in the FPUT literature a method to construct 5-wave resonances. They describe in detail the systematic construction of 5-wave resonances (also known as resonant quintets), a construction requiring $N > 6$ and $3 \mid N$ (3 divides N , i.e., N is divisible by 3). A thorough analysis of the clustering of resonant quintets via common-mode connections is presented, crucially identifying a large number of highly symmetrical clusters with octahedron topology (termed octahedron clusters). Moreover, we identify the set of all wavenumbers involved in some 5-wave resonant interaction. In **Section 9** we proceed to deepen the analysis of these octahedron clusters by analysing the inter-cluster connectivity via common modes across different octahedron clusters, a crucial aspect that provides the basis for the ideas of *thermalisation* due to nonlinear interactions in the FPUT Hamiltonian system. This defines the *super-clusters*, namely the graphs whose vertices are the octahedron clusters and whose edges represent all possible inter-cluster connections via common modes. The striking result is that the super-clusters are a disjoint union of connected subgraphs, which means that the universe of available modes separates into independent components that do not interact with each other. Via a couple of theorems we completely classify these components in terms of the set of divisors d of N such that $2 \nmid d$ and $3 \nmid d$. We analyse the super-clusters for two examples: $N = 3 \cdot 5^2 = 75$ and $N = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. In **Section 10** we discuss the implications of the super-cluster decomposition into independent pieces in regimes dominated by 5-wave resonant interactions, regarding the hypotheses of thermalisation. We show that, for general N , 6-wave interactions must be included in order to achieve a full-scale thermalisation. We conclude that the idea of component-wise thermalisation via 5-wave interactions (faster than 6-wave interactions) is interesting and deserves attention. Finally in **Section 11** we provide the solid mathematical underpinnings for the explicit construction of the general solution to the resonance conditions (2), in terms of rings and groups of polynomials with integer coefficients. We conclude with some perspectives and discussions in **Section 12**.

2. Background on the $(\alpha + \beta)$ -FPUT model

The Hamiltonian for a chain of N identical particles of mass m , subject to a force of the type in (1) can be expressed as an unperturbed Hamiltonian, H_0 , plus two perturbative terms, H_3 , H_4 :

$$H = H_0 + H_3 + H_4 \quad (3)$$

with

$$\begin{aligned} H_0 &= \sum_{j=1}^N \left(\frac{1}{2m} p_j^2 + \kappa \frac{1}{2} (q_j - q_{j+1})^2 \right), \\ H_3 &= \frac{\alpha}{3} \sum_{j=1}^N (q_j - q_{j+1})^3, \\ H_4 &= \frac{\beta}{4} \sum_{j=1}^N (q_j - q_{j+1})^4. \end{aligned} \quad (4)$$

$q_j(t)$ is the displacement of the particle j from its equilibrium position and $p_j(t)$ is the associated momentum. The canonical equations of motion are:

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} = \frac{p_j}{m}, \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} = \kappa (q_{j-1} - 2q_j + q_{j+1}) + \alpha [(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2] + \beta [(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3] \end{aligned} \quad (5)$$

with $j = 0, 1, \dots, N-1$. The Newton's equation reduces to:

$$m\ddot{q}_j = \kappa (q_{j+1} + q_{j-1} - 2q_j) + \alpha [(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2] + \beta [(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3] \quad (6)$$

This is known as the $\alpha + \beta$ -FPUT model.

2.1. Equation in Fourier space

Our approach is developed in Fourier space (periodic boundary conditions are assumed, i.e. $q_N = q_0$) and the following definitions of the direct and inverse Discrete Fourier Transform are adopted:

$$Q_k = \frac{1}{N} \sum_{j=0}^{N-1} q_j e^{-i2\pi k j/N}, \quad q_j = \sum_{k=0}^{N-1} Q_k e^{i2\pi j k/N}, \quad (7)$$

where Q_k are the Fourier amplitudes of the displacement; similarly P_k can be introduced as the Fourier transform of the momentum p_j . In terms of degrees of freedom, it will turn out that Q_0 is a real and ignorable coordinate, thus P_0 (also real) is a constant of motion. Therefore, there are $N-1$ interacting degrees of freedom (or $2N-2$ if we count P and Q independently). From here on we will use the convention that the wavenumbers k parameterizing these degrees of freedom are positive: $1 \leq k \leq N-1$. As we will see later, this convention is very useful when dealing with the frequency resonance conditions. Notice that, in this convention, the reality of q_j and p_j is equivalent to the conditions

$$Q_{N-k} = Q_k^*, \quad P_{N-k} = P_k^*, \quad k = 1, \dots, N-1. \quad (8)$$

The Poisson bracket changes from the canonical one $[q_j, p_{j'}] = \delta_{jj'}$ to:

$$[Q_k, P_{k'}^*] = \frac{1}{N^2} \sum_{j=0}^{N-1} e^{-i2\pi k j/N} \sum_{j'=0}^{N-1} e^{i2\pi k' j'/N} [q_j, p_{j'}] = \frac{1}{N^2} \sum_{j=0}^{N-1} e^{i2\pi (k'-k)j/N} = \frac{\delta_{kk'}}{N}.$$

Inserting the definition of the Fourier transform in the Hamiltonian we get:

$$\begin{aligned} \frac{H}{N} &= \frac{P_0^2}{2m} + \frac{1}{2m} \sum_{k=1}^{N-1} (|P_k|^2 + m^2 \omega_k^2 |Q_k|^2) + \frac{1}{3} \sum_{k_1, k_2, k_3=1}^{N-1} \tilde{V}_{1,2,3} Q_1 Q_2 Q_3 \delta_{1+2+3} \\ &+ \frac{1}{4} \sum_{k_1, k_2, k_3, k_4=1}^{N-1} \tilde{T}_{1,2,3,4} Q_1 Q_2 Q_3 Q_4 \delta_{1+2+3+4}, \end{aligned}$$

where we define the dispersion relation:

$$\omega_k = \omega(k) = 2\sqrt{\frac{\kappa}{m}} \sin(\pi k/N), \quad 1 \leq k \leq N-1 \quad (9)$$

(notice that ω_k is positive over our conventional range of k values), and we have introduced, in the triple and quadruple sums, the notation: $Q_i = Q(k_i, t)$, $\delta_{1+2+3} = \delta(k_1 + k_2 + k_3 \bmod N)$, which is the Kronecker δ , taking value 1 when the modular-arithmetic condition $k_1 + k_2 + k_3 = 0 \bmod N$ is satisfied. This condition arises from the spatial discreteness of the original problem. Similarly, $\delta_{1+2+3+4} = \delta(k_1 + k_2 + k_3 + k_4 \bmod N)$. Finally, $\tilde{T}_{1,2,3,4} = \tilde{T}(k_1, k_2, k_3, k_4)$ and $\tilde{V}_{1,2,3} = \tilde{V}(k_1, k_2, k_3)$ are two matrices that weigh the nonlinear interactions:

$$\tilde{V}_{1,2,3} = 8i\alpha e^{i\pi(k_1+k_2+k_3)/N} \sin \frac{\pi k_1}{N} \sin \frac{\pi k_2}{N} \sin \frac{\pi k_3}{N}, \quad (10)$$

$$\tilde{T}_{1,2,3,4} = 16\beta e^{i\pi(k_1+k_2+k_3+k_4)/N} \sin \frac{\pi k_1}{N} \sin \frac{\pi k_2}{N} \sin \frac{\pi k_3}{N} \sin \frac{\pi k_4}{N}. \quad (11)$$

The equation of motion takes the following form:

$$\ddot{Q}_1 + \omega_1^2 Q_1 = \frac{1}{m} \sum_{k_2, k_3} \tilde{V}_{1,2,3} Q_2 Q_3 \delta_{1+2+3} + \frac{1}{m} \sum_{k_2, k_3, k_4} \tilde{T}_{1,2,3,4} Q_2 Q_3 Q_4 \delta_{1+2+3+4}, \quad (12)$$

where all the sums on k_j goes from 1 to $N-1$.

2.2. Normal modes

We introduce the normal variable (or wave action variable) $a_k = a(k, t)$ via:

$$a_k = \frac{1}{\sqrt{2m\omega_k}} (m\omega_k Q_k + iP_k), \quad a_{N-k}^* = \frac{1}{\sqrt{2m\omega_k}} (m\omega_k Q_k - iP_k), \quad k = 1, \dots, N-1 \quad (13)$$

where we remark again that ω_k is positive for the selected values of k . Poisson brackets are now:

$$[a_k, a_{k'}^*] = \frac{1}{2m\omega_k} [(m\omega_k Q_k + iP_k), (m\omega_{k'} Q_{k'}^* - iP_{k'}^*)] = -\frac{i}{N} \delta_{kk'}.$$

The above relations are easy to invert to obtain:

$$Q_k = \frac{1}{\sqrt{2m\omega_k}}(a_k + a_{N-k}^*) \quad (14)$$

which can be used directly in the Hamiltonian (9) to get (we discard the P_0 term from here on):

$$\begin{aligned} \frac{H}{N} = & \sum_{k=1}^{N-1} \omega_k |a_k|^2 + \sum_{k_1, k_2, k_3} \left[\frac{1}{3} (V_{1,2,3} a_1 a_2 a_3 + c.c.) \delta_{1+2+3} + (V_{-1,2,3} a_1^* a_2 a_3 + c.c.) \delta_{1-2-3} \right] \\ & + \sum_{k_1, k_2, k_3, k_4} \left[(T_{-1,2,3,4} a_1^* a_2 a_3 a_4 + c.c.) \delta_{1-2-3-4} + \frac{3}{2} T_{-1,-2,3,4} a_1^* a_2^* a_3 a_4 \delta_{1+2-3-4} \right. \\ & \left. + \frac{1}{4} (T_{1,2,3,4} a_1 a_2 a_3 a_4 + c.c.) \delta_{1+2+3+4} \right], \end{aligned} \quad (15)$$

with

$$\begin{aligned} V_{\pm 1,2,3} &= \frac{\tilde{V}_{\pm 1,2,3}}{2m^{3/2} \sqrt{2m\omega_{k_1} \omega_{k_2} \omega_{k_3}}} = \pm \frac{i\alpha e^{i\pi(\pm k_1 + k_2 + k_3)/N}}{(m\kappa)^{3/4}} \sqrt{\sin \frac{\pi k_1}{N} \sin \frac{\pi k_2}{N} \sin \frac{\pi k_3}{N}}, \\ T_{\pm 1,2,3,4} &= \frac{\tilde{T}_{\pm 1,2,3,4}}{4m^2 \sqrt{\omega_{k_1} \omega_{k_2} \omega_{k_3} \omega_{k_4}}} = \pm \frac{\beta e^{i\pi(\pm k_1 + k_2 + k_3 + k_4)/N}}{m\kappa} \sqrt{\sin \frac{\pi k_1}{N} \sin \frac{\pi k_2}{N} \sin \frac{\pi k_3}{N} \sin \frac{\pi k_4}{N}}, \\ T_{-1,-2,3,4} &= \frac{\tilde{T}_{-1,-2,3,4}}{4m^2 \sqrt{\omega_{k_1} \omega_{k_2} \omega_{k_3} \omega_{k_4}}} = \frac{\beta e^{i\pi(-k_1 - k_2 + k_3 + k_4)/N}}{m\kappa} \sqrt{\sin \frac{\pi k_1}{N} \sin \frac{\pi k_2}{N} \sin \frac{\pi k_3}{N} \sin \frac{\pi k_4}{N}}. \end{aligned} \quad (16)$$

The equation of motion becomes

$$\begin{aligned} i \frac{\partial a_1}{\partial t} = \frac{1}{N} \frac{\delta H}{\delta a_1^*} = & \omega_{k_1} a_1 + \sum_{k_2, k_3} \left(V_{-1,2,3} a_2 a_3 \delta_{1-2-3} - 2V_{-3,1,2} a_2^* a_3 \delta_{1+2-3} - V_{1,2,3} a_2^* a_3^* \delta_{1+2+3} \right) \\ & + \sum_{k_2, k_3, k_4} \left(T_{-1,2,3,4} a_2 a_3 a_4 \delta_{1-2-3-4} + 3T_{-1,-2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4} \right. \\ & \left. + 3T_{-4,1,2,3} a_2^* a_3^* a_4 \delta_{1+2+3-4} + T_{1,2,3,4} a_2^* a_3^* a_4^* \delta_{1+2+3+4} \right). \end{aligned} \quad (17)$$

We emphasise key aspects of this equation:

1. It is equivalent to Eq. (6). The degrees of freedom are the complex modes a_k, a_k^* , $k = 1, \dots, N-1$, so a total of $2N-2$ degrees of freedom, just like originally –after separating the conserved quantity P_0 and its conjugate variable Q_0 .
2. While it might not be suitable for numerical computations, it is a very good starting point for developing theoretical approaches. In particular, all the possible mechanisms of interaction are highlighted by the presence of the Kronecker δ .

2.3. Canonical transformation and reduced Hamiltonian

Eq. (17) contains three and four wave interactions which are related to the quadratic and cubic nonlinearity contained in the Newton's equation. If one is interested in the long time behavior of the system, then it can be argued that some interactions play a more important role; indeed, nonlinearity in Fourier space results in an intrinsic forcing which is efficient if its frequency matches perfectly the frequency of oscillation of the forced wave. In terms of our equation of motion, the condition on Kronecker δ over wave numbers should be accompanied by an analogue condition in frequencies. For example, considering the last nonlinear term in the first line of Eq. (17), the resonant condition would imply that:

$$\begin{aligned} k_1 + k_2 + k_3 &= 0 \pmod{N} \\ \omega_1 + \omega_2 + \omega_3 &= 0, \end{aligned} \quad (18)$$

Three wavenumbers with $1 \leq k_j \leq N-1$ can add up to N or $2N$, therefore it is always possible to satisfy the first equation; however, it is obvious that, because frequencies are always positive, the second condition cannot be satisfied. Moreover, it is known [16] (see also Theorem 3.8 below) that all 3-wave processes are *forbidden*. Also, one can show that the leading non-trivial resonant process in Eq. (17) is the following:

$$\begin{aligned} k_1 + k_2 - k_3 - k_4 &= 0 \pmod{N} \\ \omega_1 + \omega_2 - \omega_3 - \omega_4 &= 0. \end{aligned} \quad (19)$$

This known result (see [16]), which we will prove again in Section 3.2, implies that all other terms can be removed from the Hamiltonian. Following the work [17], a suitable canonical transformation can be used to the purpose. The idea is to

impose the following near identity transformation:

$$\begin{aligned} a_1 = b_1 + \sum_{k_2, k_3} & \left(A_{1,2,3}^{(1)} b_2 b_3 \delta_{1-2-3} + A_{1,2,3}^{(2)} b_2^* b_3 \delta_{1+2-3} + A_{1,2,3}^{(3)} b_2^* b_3^* \delta_{1+2+3} \right) \\ & + \sum_{k_2, k_3, k_4} \left(B_{1,2,3,4}^{(1)} b_2 b_3 b_4 \delta_{1-2-3-4} + B_{1,2,3,4}^{(2)} b_2^* b_3 b_4 \delta_{1+2-3-4} \right. \\ & \left. + B_{1,2,3,4}^{(3)} b_2^* b_3^* b_4 \delta_{1+2+3-4} + B_{1,2,3,4}^{(4)} b_2^* b_3^* b_4^* \delta_{1+2+3+4} \right) \end{aligned} \quad (20)$$

and select the matrices $A_{1,2,3}^{(i)}$, $B_{1,2,3,4}^{(i)}$ in order to remove the unwanted interactions. Note that the transformation is asymptotic in the sense that it is formally valid only for weak nonlinearity. The algebra is quite lengthy and we refer to [17,18] for the details. As an example, the coefficients $A_{1,2,3}^{(i)}$ turn out to be given by

$$A_{1,2,3}^{(1)} = \frac{V_{1,2,3}}{\omega_3 + \omega_2 - \omega_1}, \quad A_{1,2,3}^{(2)} = \frac{2V_{1,2,3}}{\omega_3 - \omega_2 - \omega_1}, \quad A_{1,2,3}^{(3)} = \frac{V_{1,2,3}}{-\omega_3 - \omega_2 - \omega_1}. \quad (21)$$

It is interesting to underline the presence of the frequency conditions at the denominator. If these were satisfied, then the canonical transformation would diverge; consequently, only nonresonant terms can be removed (this is the well known problem of small divisors). After removing the nonresonant terms, the Hamiltonian reduces to:

$$\frac{H}{N} = \sum_{k=1}^{N-1} \omega_k |b_k|^2 + \frac{1}{2} \sum_{k_1, k_2, k_3, k_4} \bar{T}_{1,2,3,4} b_1^* b_2 b_3 b_4 \delta_{1+2-3-4} + \text{higher-order terms} \quad (22)$$

and the equation of motion becomes

$$i \frac{\partial b_1}{\partial t} = \omega_1 b_1 + \sum_{k_1, k_2, k_3, k_4} \bar{T}_{1,2,3,4} b_2^* b_3 b_4 \delta_{1+2-3-4} + \text{higher-order terms}. \quad (23)$$

This is known as the *Zakharov equation*. While removing the nonresonant terms, the canonical transformation has also the property of naturally introducing higher order wave-wave interaction terms which have been omitted in Eq. (23). In Section 3 it will be shown that the 4-wave interactions are not sufficient for spreading energy across the spectrum (see also [5,13]) and, depending on the number of interacting particles, either the 5-wave or the 6-wave resonant interactions become the dominant interaction mechanisms. We remark that the removed 3-wave interactions do not disappear from the system without leaving trace: a memory of them is contained in the new matrix $\bar{T}_{1,2,3,4}$, whose analytical form is given in Appendix A. Moreover, most of the 5-wave resonances, which exist when N is divisible by 3 and greater than 6, can be constructed starting from the 3-wave frequency resonances. We mention that the approach used to reach Eq. (23) is analogous to the one in the problem of surface gravity waves in infinite water depth, where, despite the fact that the system is continuous and the dispersion relation is different, the lowest-order resonant interaction is the four-wave one.

It is interesting to note that it is possible to show that the Hamiltonian in (22) is completely integrable if higher order terms are neglected. This important result was achieved for the $\alpha + \beta$ -FPUT in [5]. More specifically, it was shown that any periodic FPUT chain with an odd number N of particles admits a *Birkhoff normal* form up to order 4, whereas any periodic FPUT chain with N even admits a *resonant normal* form up to order 4. This resonant normal form of order 4 turns out to be completely integrable.

In the present manuscript we focus on the *existence* of exact resonances. Their role on the actual dynamics of the system will be considered elsewhere. However some explanation is in order. The original $\alpha + \beta$ -FPUT problem that we consider does not have 5- or 6-wave interactions. Those interactions may arise from the combined effective wave-wave interactions of lower order, in what is called a normal form. Those effective 5- and 6-wave interactions can be calculated via the Wyld diagrammatic technique, as detailed in [19]. For example, a sequence of three 3-wave interactions would result in effective 5-wave resonant interactions, and a sequence of four 3-wave interactions would result in effective 6-wave resonant interactions. We have checked, by direct calculations in the α -FPUT system for $N = 5$ (the lowest number of particles where irreducible 6-wave resonances are present and the normal form is not integrable) and $N = 9$ (the lowest number of particles where irreducible 5-wave resonances are present and the normal form is not integrable), that the interaction coefficients for the effective 6-wave and 5-wave resonant interactions are non-zero, thus leading to non-trivial dynamics.

3. Exact M-wave resonant interactions: Statement of the problem

3.1. Basic definitions

As mentioned an important energy transfer occurs between modes whose wavenumbers and frequencies satisfy exact resonance conditions. Here we rephrase the problem in a more precise way, suitable for analytical treatment. Consider N particles and M wavenumbers, k_1, k_2, \dots, k_M , which are integers satisfying $1 \leq k_j \leq N - 1$ for all $j = 1, \dots, M$. In principle there is no limit on the number of interacting waves M , because this is determined by the order of the nonlinear interaction

appearing in the extra terms of Eq. (23). If $M \geq N$ then the corresponding nonlinear interaction is self-interacting for some mode(s), simply because there are only $N - 1$ different wavenumbers and we are picking M of them to construct the resonance. In practical applications, however, $M = 5, 6$ will be the most relevant cases regarding thermalisation, with N arbitrary. For an M -wave nonlinear interaction, wavenumbers satisfy the momentum conservation condition

$$k_1 + \dots + k_S = k_{S+1} + \dots + k_{S+T} \pmod{N}, \quad (24)$$

where $S + T = M$ and the symbol “mod N ” means equality modulo N . This process converts S waves into T waves.

As previously mentioned, we are interested in the long time behaviour of a weakly nonlinear FPUT chain whose dynamics is ruled by exact resonant interactions. Therefore, the condition on the momentum, Eq. (24), is accompanied by the following frequency condition:

$$\omega(k_1) + \dots + \omega(k_S) = \omega(k_{S+1}) + \dots + \omega(k_{S+T}), \quad (25)$$

We recall that in the interval $1 \leq k_j \leq N - 1$ the dispersion relation is positive and it reduces to

$$\omega(k) = 2 \sin(\pi k/N), \quad (26)$$

where we have set for simplicity $m = \kappa = 1$.

Eqs. (24)–(26) with N, S, T natural numbers and $M = S + T$, determine the general problem of finding M -wave exact resonances. The unknowns are the integers k_1, \dots, k_M , satisfying $1 \leq k_j \leq N - 1$ for all $j = 1, \dots, M$. The equations are thus a system of Diophantine equations. The rest of this paper is dedicated to finding both general and algorithmic solutions to this problem. Note that the results are valid for $\alpha, \beta, (\alpha + \beta)$ or any other system with linear dispersion relation given by $\omega(k) = 2 \sin(\pi k/N)$.

Definition 3.1 (*M-wave resonance*). Let N be the number of particles of the FPUT system. An M -wave resonance is a list (i.e., a multi-set) $\{k_1, \dots, k_S; k_{S+1}, \dots, k_{S+T}\}$ with $S, T > 0, S + T = M$ and $1 \leq k_j \leq N - 1$ for all $j = 1, \dots, M$, that is a solution of the momentum conservation and frequency resonance conditions

$$\begin{aligned} k_1 + \dots + k_S &= k_{S+1} + \dots + k_{S+T} \pmod{N}, \\ \omega(k_1) + \dots + \omega(k_S) &= \omega(k_{S+1}) + \dots + \omega(k_{S+T}), \end{aligned} \quad (27)$$

where $\omega(k) = 2 \sin(\pi k/N)$.

Remark 3.2. Notice that this definition contains trivial resonances such as the 2-wave resonances $k = k, \omega(k) = \omega(k)$ valid for any wavenumber k , which do not play a dynamical role on their own, although they do play a dynamical role in the composite or reducible resonances defined below.

Remark 3.3. Any permutation of the first S wavenumbers and any permutation of the last T wavenumbers lead to the same physical resonance. Thus in many applications we will select one representative, determined by: $k_1 \leq \dots \leq k_S$ and, when possible, either $k_{S+1} \geq \dots \geq k_{S+T}$ or $k_{S+1} \leq \dots \leq k_{S+T}$.

Remark 3.4. Also, interchanging the first list (of S wavenumbers) with the second list (of T wavenumbers) leads to the same physical resonance. To deal with this symmetry we will impose, for example, $S \geq T$ when possible.

Definition 3.5 (*Irreducible M-wave resonance*). An irreducible M -wave resonance is an M -wave resonance which does not contain sub-lists that are M' -wave resonances, for any $M' < M$. In other words, there are no integers S', T' with $0 < S' < S$ and $0 < T' < T$ such that

$$\begin{aligned} k_1 + \dots + k_{S'} &= k_{S'+1} + \dots + k_{S'+T'} \pmod{N}, \\ \omega(k_1) + \dots + \omega(k_{S'}) &= \omega(k_{S'+1}) + \dots + \omega(k_{S'+T'}), \end{aligned} \quad (28)$$

where $\omega(k) = 2 \sin(\pi k/N)$.

Definition 3.6 (*Composite or reducible M-wave resonance*). An M -wave resonance is called composite or reducible if and only if it is not irreducible.

Remark 3.7. Given an M' -wave resonance and an M'' -wave resonance, it is easy to construct a composite $(M' + M'')$ -wave resonance: let $\{k_1, \dots, k_{S'}; k_{S'+1}, \dots, k_{S'+T'}\}$ be an M' -wave resonance, and let $\{\ell_1, \dots, \ell_{S''}; \ell_{S''+1}, \dots, \ell_{S''+T''}\}$ be an M'' -wave resonance. Then the following is a composite $(M' + M'')$ -wave resonance:

$$\{k_1, \dots, k_{S'}, \ell_1, \dots, \ell_{S''}; k_{S'+1}, \dots, k_{S'+T'}, \ell_{S''+1}, \dots, \ell_{S''+T''}\}.$$

As a particular case consider $M'' = 2$, i.e. the trivial irreducible 2-wave resonance $\{k; k\}$, for any choice of k . This gives the composite $(M' + 2)$ -wave resonance $\{k_1, \dots, k_{S'}, k; k_{S'+1}, \dots, k_{S'+T'}, k\}$. In general, any number of irreducible resonances can be composed in order to generate a composite resonance, since both the momentum conservation equation and the frequency resonance condition are solved by sub-sums.

From here on we will focus mainly on irreducible M -wave resonant interactions.

3.2. Forbidden M -wave processes

It is well known that 3-wave resonances do not exist. In [16] a proof is provided using number theory. Also, an explicit result from the same reference can be used to infer that 4-wave resonances must transform 2 waves into 2 waves (i.e. in our notation $S = T = 2$ is necessary).

For any number of particles N we provide a simpler proof of these statements, and a generalisation of these to the case when the number of interacting waves M is arbitrary, by using the subadditivity of the dispersion relation.

Theorem 3.8. *Resonant processes converting 1 wave to $M - 1$ waves or $M - 1$ waves to 1 wave do not exist, for any $M \neq 2$. Also, resonant processes converting 0 wave to M waves or M waves to 0 wave do not exist, for any $M > 0$.*

Proof. The function $\omega(k) = 2|\sin(\frac{\pi k}{N})|$ is strictly subadditive for $k \in \mathbb{R}$, $k \notin N\mathbb{Z}$:

$$\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2), \quad k_1, k_2 \in \mathbb{R} \setminus N\mathbb{Z}$$

(see [20–22]). Therefore, for example, resonant processes converting 2 waves into 1 wave (or vice versa) are not allowed because this would require $\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$, which is not possible. Similarly, resonant processes converting $M - 1$ waves into 1 wave (or vice versa) are not allowed because subadditivity implies

$$\omega(k_1 + \dots + k_p) < \omega(k_1) + \dots + \omega(k_p), \quad k_1, \dots, k_p \in \mathbb{R} \setminus N\mathbb{Z},$$

for any $p \geq 2$.

Also, for any number of particles N , resonant processes converting 0 wave to M waves or M waves to 0 wave do not exist. This is because the function $\omega(k) = 2|\sin(\frac{\pi k}{N})|$ is positive for $k \in \mathbb{R} \setminus N\mathbb{Z}$. \square

The case $M = 2$ converting 1 wave to 1 wave is not covered by this theorem because there is a resonance, of only one type: the so-called ‘trivial’ resonance, given by $k = k$ and $\omega(k) = \omega(k)$, for any wavenumber k . This trivial resonance does not play any dynamical role unless it is a part of a composite resonance.

Remark 3.9. As M increases, the number of allowed (i.e., not forbidden by the above analysis) M -wave resonances increases. For example, 5-wave resonances can convert $2 \leftrightarrow 3$ waves; 6-wave resonances can convert $2 \leftrightarrow 4$ and $3 \leftrightarrow 3$ waves; 7-wave resonances can convert $2 \leftrightarrow 5$ and $3 \leftrightarrow 4$ waves, and so on.

3.3. What is new about 4-wave, 5-wave and 6-wave resonances

In the next Sections we will introduce two new methods to construct M -wave resonances: pairing-off method and cyclotomic method. The appropriate method to be used depends on properties of the number of particles N and the number of waves M . The case of 4-wave resonances has been studied extensively and all solutions are known [4,5,13,16]. A crucial aspect of 4-wave resonances is that they are integrable and thus they do not produce energy mixing across the Fourier spectrum: one needs to go to higher orders. The case of 5-wave resonances is completely new and relies on the existence of cyclotomic polynomials (to be defined below). The case of 6-wave resonances is also new and both methods (pairing-off and cyclotomic) are used to construct them. It is worth mentioning that 6-wave resonances were found recently in the particular case when N is a power of two [13]. One important corollary of the constructions below will be that, for any $N \geq 4$, there exist 6-wave resonances. Therefore, we do not need to search for M -wave resonances with $M > 6$ because these will provide less relevant corrections to the system’s behaviour, in the weakly-nonlinear limit that we are considering.

4. Pairing-off method to obtain $2S$ -wave resonances converting S waves to S waves

The pairing-off method to be introduced in this Section produces the simplest type of resonances in the FPUT system: simple in its statement, its explicit construction, and its applicability to an arbitrary number of particles N . In addition to that, it plays a fundamental role in the more complex types of resonances discussed throughout the paper. For example, one of the rigorous results proven in Section 11, Corollary 11.5, states that when the number of particles N is an odd prime or a power of 2, any M -wave resonance must be of pairing-off form, and in particular M must be even. For arbitrary N , pairing-off solutions to the resonance conditions will exist along with more complex solutions based on the cyclotomic polynomials.

4.1. General pairing-off resonances

Due to the identities $\omega(k) = \omega(N - k)$, $k = 1, \dots, N - 1$, one can build immediately a set of particular solutions of Eqs. (24) and (25) provided we choose $S = T$, so the number of interacting waves $M (= 2S)$ is even. The idea is to “pair-off” incoming and outgoing waves, as follows:

$$k_{S+j} = N - k_j, \quad j = 1, \dots, S.$$

In this way, the frequency resonance condition (25) is automatically solved “by pairs” since $\omega(k_{S+j}) = \omega(k_j)$. The momentum conservation condition (24), on the other hand, leads to a single equation:

$$k_1 + \dots + k_S = \frac{N\nu}{2}, \quad (29)$$

where the integer variable ν satisfies $2S/N \leq \nu < 3S/2$ and is introduced as a parameterisation of the modular arithmetic condition in (24). The lower bound in this latter inequality follows from the fact $k_j \geq 1$, $j = 1, \dots, S$. The upper bound follows from the fact that the symmetry referred to in Remark 3.4 allows us to avoid repeated solutions by choosing a maximum of $\lfloor S/2 \rfloor$ wavenumbers with values above $\lfloor N/2 \rfloor$. Therefore we get $k_1 + \dots + k_S \leq (N-1)\lfloor S/2 \rfloor + \lfloor N/2 \rfloor(S - \lfloor S/2 \rfloor) = (\lceil N/2 \rceil - 1)\lfloor S/2 \rfloor + S\lfloor N/2 \rfloor$, so we obtain

$$\nu \leq \frac{2}{N} \left((\lceil N/2 \rceil - 1)\lfloor S/2 \rfloor + S\lfloor N/2 \rfloor \right). \quad (30)$$

For $N \geq 4$ this upper bound is easily shown to be strictly bounded from above by $3S/2$. However, the sharper version (30) of the upper bound for ν will be useful in practical search applications. For practical purposes we consider explicitly two cases:

(i) For N even and S general, the upper bound (30) reads

$$\nu \leq \frac{2}{N} \left((N/2 - 1)\lfloor S/2 \rfloor + SN/2 \right) = S + \lfloor S/2 \rfloor - \frac{2}{N}\lfloor S/2 \rfloor.$$

(ii) For N odd and S general, the upper bound (30) reads

$$\nu \leq \frac{2}{N} \left(((N+1)/2 - 1)\lfloor S/2 \rfloor + S(N-1)/2 \right) = S + \lfloor S/2 \rfloor - \frac{1}{N}(S + \lfloor S/2 \rfloor).$$

4.2. A well-known case: Pairing-off 4-wave resonances

The case $S = T = 2$, corresponding to 4-wave resonances, has been studied extensively [4,5,13,16], due to its relation with integrability and resonant Birkhoff normal forms. An important result from Rink [16] in our notation is that, for any N , any 4-wave resonance that is not composite must be of the pairing-off form.

Solutions of Eq. (29) with $\nu = 2$ exist for all N : they are just composite resonances, made out of two ‘trivial’ 2-wave resonances, parameterised as follows, in the notation of Definition 3.1:

$$\{k_1, k_2; k_2, k_1\}, \quad 1 \leq k_1 \leq k_2 \leq N-1.$$

These resonances produce nonlinear frequency shifts and do not contribute to the energy transfer between modes [13].

Solutions of Eq. (29) with $\nu = 1$ exist for N even only and are parameterised as follows:

$$\{k_1, k_2; k_3, k_4\} = \left\{ k_1, \frac{N}{2} - k_1; N - k_1, \frac{N}{2} + k_1 \right\}, \quad k_1 = 1, \dots, \lfloor N/4 \rfloor, \quad (31)$$

the momentum conservation condition reading $k_1 + k_2 = k_3 + k_4 - N$. In general, each of these 4-wave resonances or “quartets” produce energy mixing across the four modes involved, *but there is no energy mixing across different quartets*. A simple counting shows that the mode with wavenumber $N/2$ is the only one that does not interact in any quartet. As for the other modes, two cases arise: (i) If $N/2$ is an odd integer, then there are $(N/2 - 1)/2$ such quartets with 4 distinct modes each, and therefore a total of $N - 2$ modes belong to quartets. (ii) If $N/2$ is an even integer, then the case $k_1 = N/4$ in Eq. (31) becomes a degenerate quartet made out of a repeated pairing-off resonance: $\{N/4, N/4; 3N/4, 3N/4\}$, consisting of 2 modes only. The remaining $N/4 - 1$ quartets have 4 distinct modes each, giving a sub-total of $N - 4$ modes. Adding up we get again $N - 2$ modes.

4.3. A new case: Pairing-off 6-wave resonances

The case $S = T = 3$, corresponding to 6-wave interactions, was discussed for the first time in [13] in the context of wave turbulence theory. Although the cited reference looked at the case $N = 2^r$ (i.e. a power of two) only, it is easy to show that solutions exist for any N . We will make this explicit now. The key step is to evaluate the bound (30). For $S = 3, N = 6$ we get $2S/N \leq \nu \leq S + \lfloor S/2 \rfloor - \frac{2}{N}\lfloor S/2 \rfloor \Rightarrow 1 \leq \nu \leq 3 + 2/3$. Since ν is integer we get $1 \leq \nu \leq 3$.

4.3.1. Pairing-off 6-wave resonances for N odd

For N odd one must take ν even, so we need to consider the case $\nu = 2$ only. To avoid over-counting, we assume that only one wavenumber (say, k_3) can take values above $(N-1)/2$. Thus, Eq. (29) becomes $k_1 + k_2 + k_3 = N$, with two-parameter solution

$$k_3 = N - k_1 - k_2, \quad 1 \leq k_2 \leq (N-1)/2, \quad 1 \leq k_1 \leq \min(k_2, N - 2k_2).$$

The latter inequality arises from the ordering convention $1 \leq k_1 \leq k_2 \leq k_3$.

The general solution for the 6 interacting wavenumbers $\{k_1, k_2, k_3; k_4, k_5, k_6\}$ in the case N odd is thus

$$\begin{aligned} \{k_1, k_2, k_3; k_4, k_5, k_6\} &= \{k_1, k_2, N - k_1 - k_2; N - k_1, N - k_2, k_1 + k_2\}, \\ 1 \leq k_2 &\leq (N-1)/2, \quad 1 \leq k_1 \leq \min(k_2, N - 2k_2). \end{aligned} \quad (32)$$

Wavenumber momentum condition reads $k_1 + k_2 + k_3 = k_4 + k_5 + k_6 - N$.

4.3.2. Pairing-off 6-wave resonances for N even

We take $N \geq 4$ since otherwise solutions are trivial. One can choose $\nu = 1, 2$ or 3 in Eq. (29). Again, to avoid over-counting we assume only one wavenumber (say, k_3) can take values above $N/2$.

The branch $\nu = 1$ in Eq. (29), valid for $N \geq 6$, gives $k_1 + k_2 + k_3 = N/2$, with two-parameter solution

$$k_3 = \frac{N}{2} - k_1 - k_2, \quad 1 \leq k_2 \leq \frac{N}{4} - \frac{1}{2}, \quad 1 \leq k_1 \leq \min\left(k_2, \frac{N}{2} - 2k_2\right).$$

The upper bounds on the free parameters k_2 and k_1 above arise from the ordering convention $1 \leq k_1 \leq k_2 \leq k_3 \leq N - 1$. The same comment applies to the following branches.

The next two branches in Eq. (29) are valid for $N \geq 4$. The branch $\nu = 2$ gives $k_1 + k_2 + k_3 = N$, with two-parameter solution

$$k_3 = N - k_1 - k_2, \quad 1 \leq k_2 \leq \lfloor (N-1)/2 \rfloor = N/2 - 1, \quad 1 \leq k_1 \leq \min(k_2, N - 2k_2).$$

The branch $\nu = 3$ gives $k_1 + k_2 + k_3 = 3N/2$, with two-parameter solution

$$k_3 = 3N/2 - k_1 - k_2, \quad \left\lceil \frac{N}{4} + \frac{1}{2} \right\rceil \leq k_2 \leq N/2, \quad 1 \leq k_1 \leq k_2.$$

In summary, the general solution for the 6 interacting wavenumbers k_1, \dots, k_6 is:

For branch $\nu = 1$, valid for even $N \geq 6$:

$$\{k_1, k_2, k_3; k_4, k_5, k_6\} = \left\{k_1, k_2, \frac{N}{2} - k_1 - k_2; N - k_1, N - k_2, \frac{N}{2} + k_1 + k_2\right\},$$

$$1 \leq k_2 \leq \frac{N}{4} - \frac{1}{2}, \quad 1 \leq k_1 \leq \min\left(k_2, \frac{N}{2} - 2k_2\right). \quad (33)$$

(wavenumber resonance condition reads $k_1 + k_2 + k_3 = k_4 + k_5 + k_6 - 2N$).

For branch $\nu = 2$, valid for even $N \geq 4$:

$$\{k_1, k_2, k_3; k_4, k_5, k_6\} = \{k_1, k_2, N - k_1 - k_2; N - k_1, N - k_2, k_1 + k_2\},$$

$$1 \leq k_2 \leq N/2 - 1, \quad 1 \leq k_1 \leq \min(k_2, N - 2k_2). \quad (34)$$

(wavenumber resonance condition reads $k_1 + k_2 + k_3 = k_4 + k_5 + k_6 - N$).

For branch $\nu = 3$, valid for even $N \geq 4$:

$$\{k_1, k_2, k_3; k_4, k_5, k_6\} = \{k_1, k_2, 3N/2 - k_1 - k_2; N - k_1, N - k_2, k_1 + k_2 - N/2\},$$

$$\left\lceil \frac{N}{4} + \frac{1}{2} \right\rceil \leq k_2 \leq N/2, \quad 1 \leq k_1 \leq k_2. \quad (35)$$

(wavenumber resonance condition reads $k_1 + k_2 + k_3 = k_4 + k_5 + k_6$).

More generally, the case $S \geq 4$ has solutions for any $N \geq 4$. In principle, these can be constructed explicitly as a multi-parameter solution of Eq. (29).

5. The need for a more general method illustrated with the case $N = 6$

In the case when N (number of particles) is arbitrary, the pairing-off solutions to the $2L$ -wave resonant conditions (with $2L \geq 6$) do not exhaust all possible solutions. For example, in the case $N = 6$ (six particles) there is a 6-wave resonance that is not of pairing-off form:

$$1 + 1 + 5 + 5 = 3 + 3 \pmod{6}, \quad \omega(1) + \omega(1) + \omega(5) + \omega(5) = \omega(3) + \omega(3).$$

The frequency resonance condition is satisfied because of the identity

$$\omega(1) + \omega(5) = \omega(3), \quad \text{or} \quad \sin \pi/6 + \sin 5\pi/6 = \sin 3\pi/6,$$

which is reminiscent of a triad resonance. What is the origin of this identity? The answer is given in terms of the $2N$ th root of unity and the so-called cyclotomic polynomials.

6. Writing the resonance conditions in terms of real polynomials on the $(2N)$ th root of unity

Going back to the dimensionless definition of dispersion relation, Eq. (26), it will be useful to write it as complex exponential:

$$\omega(k) \equiv -i(\zeta^k - \zeta^{-k}),$$

where

$$\zeta = \exp\left(\frac{i\pi}{N}\right) \quad (36)$$

is a primitive $2N$ th root of unity: $\zeta^{2N} = 1$.

In terms of ζ , the frequency resonance condition (25) is

$$(\zeta^{k_1} - \zeta^{-k_1}) + \dots + (\zeta^{k_S} - \zeta^{-k_S}) = (\zeta^{k_{S+1}} - \zeta^{-k_{S+1}}) + \dots + (\zeta^{k_{S+T}} - \zeta^{-k_{S+T}}). \quad (37)$$

Recalling that k_j are all positive and that ζ is a unit complex number, it follows that ζ^{-k_j} is the complex conjugate of ζ^{k_j} , so relation (37) is equivalent to the statement that a polynomial is real:

$$\rho(\zeta) \equiv \zeta^{k_1} + \dots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \dots + \zeta^{k_{S+T}}) \in \mathbb{R}. \quad (38)$$

In other words, solving the frequency resonance conditions is an easy task: it amounts to finding all real polynomials on the variable ζ . Before tackling the general problem we present two examples:

Example 6.1. The pairing-off resonances shown in a previous section correspond to a “pairing-off” real polynomial, made out of real binomials: by setting $S = T$ and $k_{S+j} = N - k_j$ we obtain, from Eq. (38),

$$\rho(\zeta) \equiv \zeta^{k_1} + \dots + \zeta^{k_S} - (\zeta^{N-k_1} + \dots + \zeta^{N-k_S}) = (\zeta^{k_1} + \zeta^{-k_1}) + \dots + (\zeta^{k_S} + \zeta^{-k_S}),$$

which is real, pair by pair.

Example 6.2. The resonance shown in the previous section does not correspond to the pairing-off form. Rather, it corresponds to an element of the kernel of map (38), since, for $N = 6$ we have

$$\zeta = \exp\left(\frac{i\pi}{6}\right) \implies \zeta + \zeta^5 - \zeta^3 = 0.$$

Thus, the resonance corresponds to $S = 4$, $T = 2$ with wavenumbers $k_1 = k_2 = 1$, $k_3 = k_4 = 5$, and $k_5 = k_6 = 3$ so we obtain, from Eq. (38),

$$\rho(\zeta) \equiv \zeta^{k_1} + \zeta^{k_2} + \zeta^{k_3} + \zeta^{k_4} - \zeta^{k_5} - \zeta^{k_6} = 2(\zeta^{k_1} + \zeta^{k_3} - \zeta^{k_5}) = 0,$$

again real.

Definition 6.3 (Real FPUT polynomial). For any number of particles N , a real FPUT polynomial is a polynomial $\rho(x)$ of degree $N - 1$ in the complex variable x , with integer coefficients:

$$\rho(x) = \sum_{k=1}^{N-1} \rho_k x^k, \quad \rho_k \in \mathbb{Z}, \quad (39)$$

such that $\rho(\zeta) \in \mathbb{R}$, where $\zeta = \exp(i\pi/N)$.

Remark 6.4. A real polynomial may have a nontrivial constant term: $\rho(0) \neq 0$, but the constant term does not contribute to the wave resonance condition. So, for our present purposes, we can always discard the constant term from the real FPUT polynomials.

In general, while this interpretation in terms of real polynomials effectively solves the frequency resonance condition (37), one still has to solve the momentum conservation condition (24), but this is a linear problem so it is much simpler. It turns out we can re-write the problem of finding M -wave resonances entirely in terms of real FPUT polynomials.

Definition 6.5 (Resonant FPUT polynomial). A resonant FPUT polynomial is a real FPUT polynomial $\rho(x)$ that satisfies $\rho'(1) = 0 \pmod{N}$, where prime denotes differentiation.

Theorem 6.6 (M -wave resonances in terms of resonant FPUT polynomials). For any number of particles N , the problem of finding M -wave resonances, Eq. (27), is equivalent to the problem of finding a resonant FPUT polynomial $\rho(x)$, i.e. a polynomial $\rho(x)$ of degree $N - 1$ in the complex variable x , with integer coefficients, such that

$$\rho(\zeta) \in \mathbb{R} \quad \text{and} \quad \rho'(1) = 0 \pmod{N},$$

where $\zeta = \exp(i\pi/N)$ and prime denotes differentiation. Moreover, writing the resonant FPUT polynomial as $\rho(x) = \sum_{k=1}^{N-1} \rho_k x^k$ we obtain the number M of resonant waves via the formula

$$M = \sum_{k=1}^{N-1} |\rho_k|. \quad (40)$$

Proof. First, the frequency resonance condition in (27) is equivalent to condition (38), which in turn can be re-written in terms of the real FPUT polynomial (39) as

$$\rho(\zeta) \equiv \sum_{k=1}^{N-1} \rho_k \zeta^k \in \mathbb{R}, \quad (41)$$

where the integers $\{\rho_k\}_{k=1}^{N-1}$ are defined by

$$\rho_k = \sum_{j=1}^S \delta_k^{k_j} - \sum_{j=S+1}^{S+T} \delta_k^{k_j}, \quad 1 \leq k \leq N-1. \quad (42)$$

Notice that in composite resonances where one of the components is a ‘trivial’ 2-wave resonance, there will be cancelling contributions from the two sums in Eq. (42). Thus, we will concentrate on the case where no trivial resonances are included in the composite resonances.

Second, the momentum conservation condition in (27) can be written in the form

$$\left(\frac{d\rho(x)}{dx} \right) \Big|_{x=1} = \sum_{k=1}^{N-1} k \rho_k = 0 \pmod{N}, \quad (43)$$

where again $\{\rho_k\}_{k=1}^{N-1}$ are given in (42).

Conversely, suppose given a real FPUT polynomial: $\rho(x) = \sum_{k=1}^{N-1} \rho_k x^k$, $\rho_k \in \mathbb{Z}$. Then $\sum_{k=1}^{N-1} \rho_k \zeta^k = \sum_{k=1}^{N-1} \rho_k \zeta^{-k}$ and hence

$$\sum_{k=1}^{N-1} \rho_k (\zeta^k - \zeta^{-k}) = 0$$

gives a solution to the wave resonance condition in (27), with $M = \sum_{k=1}^{N-1} |\rho_k|$ (to be shown below). The momentum conservation condition (43) remains to be discussed.

In terms of the new notation, the list (i.e. the set that may include repeated elements) of S ‘incoming’ momenta is defined by

$$\{k_1, \dots, k_S\} = \{k | 1 \leq k \leq N-1, \rho_k > 0\}.$$

In order to avoid over-counting, we order these as $k_1 \leq \dots \leq k_S$.

The list of T ‘outgoing’ momenta is defined similarly by

$$\{k_{S+1}, \dots, k_{S+T}\} = \{k | 1 \leq k \leq N-1, \rho_k < 0\}.$$

Again in order to avoid over-counting, it is useful to order these as $k_{S+1} \geq \dots \geq k_{S+T}$.

The momentum conservation condition is

$$k_1 + \dots + k_S = k_{S+1} + \dots + k_{S+T} \pmod{N}$$

(all momenta are integers between 1 and $N-1$). Alternatively, this can be constructed directly from the polynomial $\rho(x)$ as follows: write

$$\rho'(1) = 0 \pmod{N},$$

which gives

$$\sum_{k=1}^{N-1} k \rho_k = 0 \pmod{N}.$$

The interpretation of this is that the positive coefficients provide us the incoming momenta (with multiplicities ρ_k) and the negative terms provide us the outgoing momenta. Thus the number S of ‘incoming’ waves and the number T of ‘outgoing’ waves can be computed from the integer coefficients ρ_k of the real FPUT polynomial, as follows:

$$S = \frac{1}{2} \sum_{k=1}^{N-1} |\rho_k| + \rho_k, \quad T = \frac{1}{2} \sum_{k=1}^{N-1} |\rho_k| - \rho_k.$$

Finally, the total number of waves is given by $M = S + T = \sum_{k=1}^{N-1} |\rho_k|$. \square

So we have reduced the general problem of finding M -wave resonances to the problem of finding resonant FPUT polynomials. Solving the frequency resonance condition (41) amounts to finding a basis for the real FPUT polynomials, which is a relatively easy task that requires however a number of mathematical constructions, to be discussed in Section 11. The momentum conservation condition (43) becomes a linear Diophantine problem on the integer coefficients $\{\rho_k\}_{k=1}^{N-1}$.

Remark 6.7. To illustrate how easy it is to find resonances, we stress that if $\rho(x)$ is any real FPUT polynomial then the frequency condition is automatically satisfied. A solution of the momentum condition can be found by defining $d = \gcd(N, \sum_{k=1}^{N-1} k \rho_k)$ so the new polynomial $(N/d)\rho(x)$ is a resonant FPUT polynomial since it satisfies, in addition, the M -wave momentum condition, with $M = \frac{N}{d} \sum_{k=1}^{N-1} |\rho_k|$.

The above remark shows that we can find resonances. Now, when the number of particles N is arbitrary, there are two types of questions that we address in the following Sections:

1. Algorithmic constructions (Sections 7–10 and Appendix B):
 - (a) 5-wave resonances, necessarily not of pairing-off form because the number of interacting waves ($M = 5$) is odd.
 - (b) 6-wave resonances that are not of pairing-off form.
 - (c) Connectivity of resonant clusters when combining 4-, 5- and 6-wave resonances, and their implications regarding the dynamics of the FPUT system and its eventual thermalisation.
2. Mathematical underpinnings (Section 11 and Appendix C):
 - (a) Construction of a basis for the real FPUT polynomials.
 - (b) Full understanding of a real FPUT polynomial as a sum of pairing-off terms plus terms that are not of pairing-off form (i.e., based exclusively on cyclotomic polynomials).
 - (c) General results on the total number of interacting waves M .

7. The cyclotomic method: constructing resonant FPUT polynomials of short length based on cyclotomic polynomials

The results of this Section are motivated by the rigorous construction of the basis for the real FPUT polynomials (Section 11.2.2). However this Section stands alone.

Definition 7.1 (Cyclotomic polynomial). For any $n \in \mathbb{N}$, the cyclotomic polynomial $\Phi_n(x)$ is the unique irreducible polynomial with integer coefficients which divides the polynomial $x^n - 1$ but does not divide the polynomial $x^j - 1$ for any $j < n$. One has $\Phi_n(z_n) = 0$, where $z_n \equiv \exp\left(\frac{2i\pi}{n}\right)$. More generally, $\Phi_n(\exp\left(\frac{2i\pi k}{n}\right)) = 0$ for any integer k such that $1 \leq k < n$ and $\gcd(k, n) = 1$.

The key ingredient to construct M -wave resonances stems from the fact that if an odd prime p is such that $p|N$ (in words, if p divides N) then the polynomial

$$g_p(x) \equiv \Phi_p(x^{2N/p}) = 1 + x^{2N/p} + x^{2 \cdot (2N/p)} + \dots + x^{(p-1) \cdot (2N/p)}$$

is a “real” polynomial in the sense that, evaluating at $x = \zeta$, it vanishes because $\Phi_p(e^{2i\pi/p}) = 0$. However, this polynomial is not yet a real FPUT polynomial, because its degree is greater than $N - 1$. But one can reduce its degree to $N - 1$ by repeated use of the identity $\zeta^N = -1$: if the coefficient of x^k is nonzero and if $k = tN + r$, with $r \leq N - 1$, then replace x^k with $(-1)^t x^r$. This leads to the following real FPUT polynomial:

$$f_p(x) = 1 - x^{N/p} + x^{2N/p} - \dots + x^{(p-1)N/p}.$$

In fact, this polynomial is in the kernel of the map (38), i.e. $f_p(\zeta) = 0$. This allows us to generate a number of real FPUT polynomials. First, multiply $f_p(x)$ by x^n with $1 \leq n \leq N/p - 1$, producing the set of real FPUT polynomials

$$P_n^{(p)}(x) \equiv x^n f_p(x) = x^n - x^{N/p+n} + x^{2N/p+n} - \dots + x^{(p-1)N/p+n}, \quad 1 \leq n \leq N/p - 1.$$

It is remarkable that these polynomials produce automatically the frequency resonance conditions corresponding to p -wave resonances:

$$\omega(n) - \omega(N/p + n) + \omega(2N/p + n) - \dots + \omega((p-1)N/p + n) = 0, \quad 1 \leq n \leq N/p - 1,$$

which follows directly from the identity $P_n^{(p)}(\zeta) = 0$. The question to be investigated is therefore whether these polynomials satisfy the momentum conservation condition $\left. \frac{dP_n^{(p)}}{dx} \right|_{x=1} = 0 \pmod{N}$ for some n , and if the answer is no, then we need to ask whether adding another real FPUT polynomial to $P_n^{(p)}(x)$ can produce a polynomial that solves also the momentum conservation condition, i.e. a resonant FPUT polynomial.

To illustrate the idea consider the case $p = 3$ where $3|N$. Then all the polynomials

$$P_n^{(3)}(x) = x^n f_3(x) = x^n - x^{N/3+n} + x^{2N/3+n}, \quad 1 \leq n \leq N/3 - 1$$

are real FPUT polynomials, i.e. they satisfy automatically the frequency resonance conditions. However, none of these polynomials can be resonant because the number M of interacting waves, which is equal to the sum of the absolute values of the coefficients of the FPUT polynomial as follows from Eqs. (39) and (40), is equal to 3 by inspection. Thus, resonance would imply the existence of 3-wave resonances, which we know is false from Theorem 3.8. It turns out that in order to produce a resonant FPUT polynomial out of $P_n^{(3)}(x)$ a simple recipe is needed: to add to it a pairing-off term of the form $x^q - x^{N-q}$, where $1 \leq q \leq N - 1$ and q is to be found explicitly. So we get the 5-wave resonances (also known as “resonant quintets”) out of the FPUT resonant polynomials

$$R_{n,q}^{(3)}(x) = x^n - x^{N/3+n} + x^{2N/3+n} + x^q - x^{N-q}, \quad 1 \leq n \leq N/3 - 1, \quad 1 \leq q \leq N - 1$$

where the frequency resonance is automatically satisfied since $R_{n,q}^{(3)}(\zeta) \in \mathbb{R}$, and the momentum conservation condition

$$\left. \frac{dR_{n,q}^{(3)}}{dx} \right|_{x=1} = 0 \pmod{N} \text{ is a Diophantine equation that is linear in } N, n \text{ and } q:$$

$$n + N/3 + 2q = 0 \pmod{N}, \quad 1 \leq n \leq N/3 - 1, \quad 1 \leq q \leq N - 1. \quad (44)$$

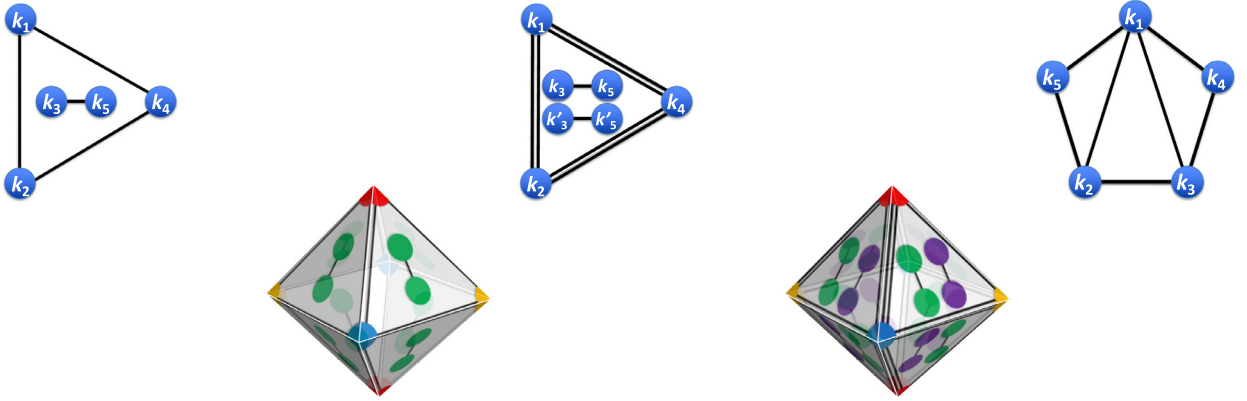


Fig. 1. Colour online. Schematic view of resonant quintets. Top left panel: A resonant quintet based on cyclotomic polynomials of length 3. The plot encodes uniquely the relations $\omega(k_1) + \omega(k_2) = \omega(k_4)$ (triad), $k_5 = N - k_3$ (pair-off terms) and $k_1 + k_2 + k_3 = k_4 + k_5 \pmod{N}$ (quintet). Top centre panel: When $6|N$ the resonant quintets depicted on the left panel appear now in pairs, with 3-common-mode connections. Notably, the inner pair-off terms satisfy the quartet resonance $k_3 + k'_5 = k'_3 + k_5 \pmod{N}$, which shows how 4-wave resonances are embedded in the 5-wave resonances. Top right panel: A resonant quintet based on cyclotomic polynomials of length 5. The plot encodes the relations $k_1 + k_2 + k_3 = k_4 + k_5 \pmod{N}$ and $\omega(k_1) + \omega(k_2) + \omega(k_3) = \omega(k_4) + \omega(k_5) \pmod{N}$. Bottom left panel: A typical FPUT resonant cluster (octahedron) made out of 8 of the simple quintets that are based on cyclotomic polynomials of length 3, depicted in the top left panel. Bottom right panel: In the case $6|N$ the typical FPUT resonant clusters are octahedrons made out of 8 of the “double” quintets depicted in the top centre panel.

In the alternative notation in terms of wavenumbers these resonances are denoted by $\{n, 2N/3 + n, q; N/3 + n, N - q\}$ as is evident from inspection of the monomials in $R_{n,q}^{(3)}(x)$ above.

One could repeat this idea by noting that each term in $P_n^{(3)}(x)$ could be replaced by its pairing-off partner, which would produce a new real FPUT polynomial of length 3. For example, consider instead of $P_n^{(3)}(x)$ the polynomial $P_n^{(3)}(x) + (x^{N-(N/3+n)} - x^{N/3+n}) = x^n - x^{2N/3-n} + x^{2N/3+n}$, and thus the 5-wave resonances come from FPUT resonant polynomials of the form $x^n - x^{2N/3-n} + x^{2N/3+n} + x^{q'} - x^{N-q'}$. The frequency resonance condition is trivially satisfied, since these are real FPUT polynomials. The momentum conservation condition reads

$$3n + 2q' = 0 \pmod{N}, \quad 1 \leq n \leq N/3 - 1, \quad 1 \leq q' \leq N - 1. \quad (45)$$

How many different cases do we have? There are 2 choices of wavenumbers for each of the 3 terms in $P_n^{(3)}(x)$, leading to a total of $2^3 = 8$ possible sets of 5-wave resonances constructed in this way, each with a relation analogous to (44) and (45). Amongst these 8 sets of 5-wave resonances, some cases will be related by symmetries so we need to be aware of these to avoid over-counting. In the next Section we present the explicit solutions of these equations, leading to a family of interconnected 5-wave resonances.

Finally, these ideas could be extended to the case $p = 5$ where one could ask again whether the polynomial $P_n^{(5)}(x)$ is resonant or not, in order to obtain more 5-wave resonances. The explicit solution is shown in the next Section as well.

We leave for Appendix B the explicit solutions for 6-wave resonances that are not of pairing-off form. These new solutions are built by either adding two real FPUT polynomials of length 3 in order to construct a resonant FPUT polynomial of length 6, or by adding a real FPUT polynomial of length 3 with one of length 5, in such a way that one of the monomials from each polynomial cancel out, leading to a resonant FPUT polynomial of length 6 again.

8. Constructing 5-wave resonances for arbitrary $N (> 6)$

We provide results for 5-wave resonances, also called resonant quintets, in the notation $\{k_1, k_2, k_3; k_4, k_5\}$ that was introduced earlier.

First, we provide four results stemming from the existence of cyclotomic polynomials of length 3. By construction, these resonant quintets satisfy the following relations:

- (i) The 3-wave frequency resonance $\omega(k_1) + \omega(k_2) = \omega(k_4)$.
- (ii) The pairing-off frequency relation $\omega(k_3) = \omega(k_5)$, stemming from $k_5 = N - k_3$.

Because of these structural relations, one can visualise these quintets as in Fig. 1, top left panel (applicable to the case $3|N$ and $6 \nmid N$). Fig. 1, top centre panel, refers to the case $6|N$, where the quintets “double”, i.e. they come in pairs connected via 3 common modes. The reason for this doubling can be seen by looking at Eq. (44): if q is a solution then so is $q + N/2 \pmod{N}$.

8.1. Case 0: $3 \nmid N$.

In this case there are no 5-wave resonances from our method.

8.2. Case 1 (simple octahedra): $3|N$ and $6 \nmid N$.

We obtain that the resonant quintets are grouped in $\lfloor \frac{N}{6} \rfloor$ octahedron clusters of 8 quintets each, with the property that, within each cluster, any quintet is connected to other 4 quintets via at least 2-common-mode connections and to other 3 quintets via at least 1-common-mode connections. See Fig. 1 (top left panel and bottom left panel) for a depiction of a quintet and an 8-quintet cluster (octahedron). The quintets within each octahedron cluster are given explicitly in terms of the even parameter n that runs from 2 to $\frac{N}{3} - 1$:

$$\begin{aligned} & \left\{ n, \frac{2N}{3} + n, \frac{N}{3} - \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} + \frac{n}{2} \right\} \\ & \left\{ n, \frac{N}{3} - n, \frac{n}{2}; \frac{N}{3} + n, N - \frac{n}{2} \right\} \\ & \left\{ n, \frac{2N}{3} + n, N - \frac{3n}{2}; \frac{2N}{3} - n, \frac{3n}{2} \right\} \\ & \left\{ n, \frac{N}{3} - n, \frac{2N}{3} - \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{3} + \frac{n}{2} \right\} \\ & \left\{ N - n, \frac{2N}{3} + n, \frac{N}{3} + \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} - \frac{n}{2} \right\} \\ & \left\{ N - n, \frac{N}{3} - n, N - \frac{3n}{2}; \frac{N}{3} + n, \frac{3n}{2} \right\} \\ & \left\{ N - n, \frac{2N}{3} + n, N - \frac{n}{2}; \frac{2N}{3} - n, \frac{n}{2} \right\} \\ & \left\{ N - n, \frac{N}{3} - n, \frac{2N}{3} + \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{3} - \frac{n}{2} \right\} \\ & n = 2, \dots, \frac{N}{3} - 1 \quad (n \text{ even}). \end{aligned} \quad (46)$$

Except for rare degenerate cases, the quintets within each octahedron cluster are connected via 2 common modes: this is due to the connectivity of the triads (k_1, k_2, k_4) that form the core of the quintets. Explicitly, the wavenumbers belonging to these core triads correspond to the vertices of the octahedron and appear in three “conjugate” pairs: $(n, N - n)$, $(n + N/3, 2N/3 - n)$ and $(n + 2N/3, N/3 - n)$. A conjugate pair corresponds to opposite vertices (in the picture in Fig. 1, bottom left panel, they have the same colour for identification purposes). The colouring of the pair-off modes depicted on the faces is for reference purposes only: while on any given face the wavenumbers are in pair-off form $(k_3, k_5 = N - k_3)$, and opposite faces have the same wavenumbers, for non-opposite faces the wavenumbers are different.

Non-interacting modes in Case 1 ($3|N \wedge 6 \nmid N$). A direct inspection of Eq. (46) shows that all wavenumbers in the range $1 \leq k \leq N - 1$ appear, with the only exception of the two wavenumbers $N/3, 2N/3$, which do not appear in any octahedron cluster when $9 \nmid N$. When $9|N$, these wavenumbers appear in the cluster generated by $n = 2N/9$ in Eq. (46).

8.3. Case 2 (double octahedra): $6|N$ and $12 \nmid N$.

In this case, due to the appearance of “double quintets” (explained at the beginning of Section 8), the clusters double in size (i.e. each cluster has now 16 quintets) and correspondingly the number of clusters is halved: there are $\frac{1}{2}(\frac{N}{6} - 1)$ clusters now. See Fig. 1 (top centre panel and bottom right panel) for a depiction of a “doubled” quintet and a 16-quintet cluster (“doubled” octahedron). The quintets within each cluster are given explicitly in terms of the even parameter n that runs from 2 to $\frac{N}{6} - 1$:

$$\begin{aligned} & \left\{ n, \frac{2N}{3} + n, \frac{N}{3} - \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} + \frac{n}{2} \right\} \quad \left\{ n, \frac{2N}{3} + n, \frac{5N}{6} - \frac{n}{2}; \frac{N}{3} + n, \frac{N}{6} + \frac{n}{2} \right\} \\ & \left\{ n, \frac{N}{3} - n, \frac{n}{2}; \frac{N}{3} + n, N - \frac{n}{2} \right\} \quad \left\{ n, \frac{N}{3} - n, \frac{N}{2} + \frac{n}{2}; \frac{N}{3} + n, \frac{N}{2} - \frac{n}{2} \right\} \\ & \left\{ n, \frac{2N}{3} + n, N - \frac{3n}{2}; \frac{2N}{3} - n, \frac{3n}{2} \right\} \quad \left\{ n, \frac{2N}{3} + n, \frac{N}{2} - \frac{3n}{2}; \frac{2N}{3} - n, \frac{N}{2} + \frac{3n}{2} \right\} \\ & \left\{ n, \frac{N}{3} - n, \frac{2N}{3} - \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{3} + \frac{n}{2} \right\} \quad \left\{ n, \frac{N}{3} - n, \frac{N}{6} - \frac{n}{2}; \frac{2N}{3} - n, \frac{5N}{6} + \frac{n}{2} \right\} \\ & \left\{ N - n, \frac{2N}{3} + n, \frac{N}{3} + \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} - \frac{n}{2} \right\} \quad \left\{ N - n, \frac{2N}{3} + n, \frac{5N}{6} + \frac{n}{2}; \frac{N}{3} + n, \frac{N}{6} - \frac{n}{2} \right\} \\ & \left\{ N - n, \frac{N}{3} - n, N - \frac{3n}{2}; \frac{N}{3} + n, \frac{3n}{2} \right\} \quad \left\{ N - n, \frac{N}{3} - n, \frac{N}{2} - \frac{3n}{2}; \frac{N}{3} + n, \frac{N}{2} + \frac{3n}{2} \right\} \\ & \left\{ N - n, \frac{2N}{3} + n, N - \frac{n}{2}; \frac{2N}{3} - n, \frac{n}{2} \right\} \quad \left\{ N - n, \frac{2N}{3} + n, \frac{N}{2} - \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{2} + \frac{n}{2} \right\} \end{aligned}$$

$$\left\{ N-n, \frac{N}{3}-n, \frac{2N}{3}+\frac{n}{2}; \frac{2N}{3}-n, \frac{N}{3}-\frac{n}{2} \right\} \quad \left\{ N-n, \frac{N}{3}-n, \frac{N}{6}+\frac{n}{2}; \frac{2N}{3}-n, \frac{5N}{6}-\frac{n}{2} \right\}$$

$$n = 2, \dots, \frac{N}{6} - 1 \quad (n \text{ even}). \quad (47)$$

Non-interacting modes in Case 2 ($6|N \wedge 12 \nmid N$). By inspection of Eq. (47), all wavenumbers in the range $1 \leq k \leq N-1$ appear, with the following exceptions:

- (i) The four wavenumbers $N/6$, $N/3$, $2N/3$, $5N/6$ do not appear in any double octahedron cluster when $9 \nmid N$. When $9|N$, these wavenumbers appear in the cluster generated by $n = N/9$ in Eq. (47).
- (ii) The wavenumber $N/2$ does not appear in any double octahedron cluster in Eq. (47).

8.4. Case 3 (squashed octahedron): $12|N$.

Here, like in Case 2, the clusters are formed out of 16 quintets. In fact, exactly the same formulae as in Eq. (47) hold, except that the even parameter n runs from 2 to $N/6$ now. However, the cluster corresponding to $n = N/6$, as can be easily checked by looking at the formulae, has many quintets repeated (in other words it is degenerate), so in the end that special cluster has 3 different pairs of quintets only. These correspond to lines 3, 4, and 5 in Eq. (47) and they are, explicitly:

$$\left\{ \frac{N}{6}, \frac{5N}{6}, \frac{3N}{4}; \frac{N}{2}, \frac{N}{4} \right\} \quad \left\{ \frac{N}{6}, \frac{5N}{6}, \frac{N}{4}; \frac{N}{2}, \frac{3N}{4} \right\}$$

$$\left\{ \frac{N}{6}, \frac{N}{6}, \frac{7N}{12}; \frac{N}{2}, \frac{5N}{12} \right\} \quad \left\{ \frac{N}{6}, \frac{N}{6}, \frac{N}{12}; \frac{N}{2}, \frac{11N}{12} \right\}$$

$$\left\{ \frac{5N}{6}, \frac{5N}{6}, \frac{5N}{12}; \frac{N}{2}, \frac{7N}{12} \right\} \quad \left\{ \frac{5N}{6}, \frac{5N}{6}, \frac{11N}{12}; \frac{N}{2}, \frac{N}{12} \right\}. \quad (48)$$

It is difficult to picture this degenerate cluster in the simple way we did for the octahedron clusters in Fig. 1. While non-degenerate octahedron clusters are constructed out of 8 triads, this degenerate cluster is based on *three triads* only, which are generated by *three wavenumbers* only: $N/6$, $5N/6$ and $N/2$. Thus, the picture is like a “squashed” octahedron.

Non-interacting modes in Case 3 ($12|N$). By inspection of Eqs. (47) and (48), all wavenumbers in the range $1 \leq k \leq N-1$ appear, with the exception of the two wavenumbers $N/3$, $2N/3$, which do not appear in any double octahedron cluster when $9 \nmid N$. When $9|N$, these wavenumbers appear in the cluster generated by $n = N/9$ in Eq. (47).

The four cases discussed above are exclusive and together they cover all possible quintets that are based on cyclotomic polynomials of length 3.

Finally, we provide one result stemming from the existence of cyclotomic polynomials of length 5. Notice that this extra result can be used in combination with the above Cases 1, 2, and 3.

8.5. Case 4 (pentagons): $15|N$.

Here there are just two extra quintets. These are:

$$\left\{ \frac{N}{15}, \frac{7N}{15}, \frac{13N}{15}; \frac{2N}{3}, \frac{11N}{15} \right\} \quad \left\{ \frac{2N}{15}, \frac{8N}{15}, \frac{14N}{15}; \frac{4N}{15}, \frac{N}{3} \right\}. \quad (49)$$

By inspection these two quintets have no common modes (but they may have common modes with the octahedra found in Cases 1, 2, 3). To picture these quintets we refer to the pentagons in Fig. 1, top right panel.

Non-interacting modes in Case 4 ($15|N$), in combination with the other cases. The wavenumbers $N/3$, $2N/3$ appear once each in the pentagon clusters of Eq. (49). As we recall from Cases 1, 2, and 3, these wavenumbers were exceptional. Thus, we summarise three combined cases:

(Case 4–1) When $15|N \wedge 2 \nmid N$, all wavenumbers in the range $1 \leq k \leq N-1$ appear in some cluster.

(Case 4–2) When $30|N \wedge 4 \nmid N$, the wavenumber $N/2$ does not appear at all. All other wavenumbers in the range $1 \leq k \leq N-1$ appear in some cluster, except for the wavenumbers $N/6$, $5N/6$, which appear if and only if $9|N$.

(Case 4–3) When $60|N$, all wavenumbers in the range $1 \leq k \leq N-1$ appear in some cluster.

We have checked by brute-force numerical search up to $N = 1155 (= 3 \cdot 5 \cdot 7 \cdot 11)$ that the cases above (case 1 to case 4) comprise all resonant quintets. Table 1 summarises the results.

9. Super-clusters: Remarks about the connectivity of the clusters of 5-wave resonances

The clusters found in the previous Section, depicted in Fig. 1 (bottom panels), appear simple when considered separately. The basic octahedron cluster, made out of 8 quintets (Eq. (46)), contains in the generic case a total of 14 different modes, a count that is broken down as follows: the 6 vertices contain 6 different modes; the 8 faces contain 2 modes each but opposite faces have the same modes, so the faces contribute with $4 \times 2 = 8$ modes, giving $6 + 8 = 14$ modes overall. The “doubled” octahedron, made out of 16 quintets (Eq. (47)), contains in the generic case a total of 22 different modes: 6 modes from the vertices plus $4 \times 4 = 16$ modes from the faces, where now each face contributes with 4 different modes but

Table 1Summary of cases of 5-wave resonances for $N > 6$, regarding the counting of octahedron clusters and total number of quintets.

$N > 6$	$5 \nmid N$	$5 N$
$3 \nmid N$	No Quintets. $\lfloor \frac{N}{6} \rfloor$ Clusters: 8 Quintets Each;	No Quintets. 1 Extra Cluster: 2 Quintets;
$3 N \wedge 6 \nmid N$	Formulae: Eq. (46). Total: $8 \lfloor \frac{N}{6} \rfloor$ Quintets. $\frac{1}{2}(\frac{N}{6} - 1)$ Clusters: 16 Quintets Each;	Formulae: Eqs. (46) and (49). Total: $8 \lfloor \frac{N}{6} \rfloor + 2$ Quintets. 1 Extra Cluster: 2 Quintets;
$6 N \wedge 12 \nmid N$	Formulae: Eq. (47). Total: $8(\frac{N}{6} - 1)$ Quintets. $\frac{N}{12} - 1$ Clusters: 16 Quintets Each;	Formulae: Eqs. (47) and (49). Total: $8(\frac{N}{6} - 1) + 2$ Quintets. 1 Extra Cluster: 2 Quintets;
$12 N$	1 Cluster: 6 Quintets; Formulae: Eqs. (47) and (48). Total: $16(\frac{N}{12} - 1) + 6$ Quintets.	Formulae: Eqs. (47), (48) and (49). Total: $16(\frac{N}{12} - 1) + 6 + 2$ Quintets.

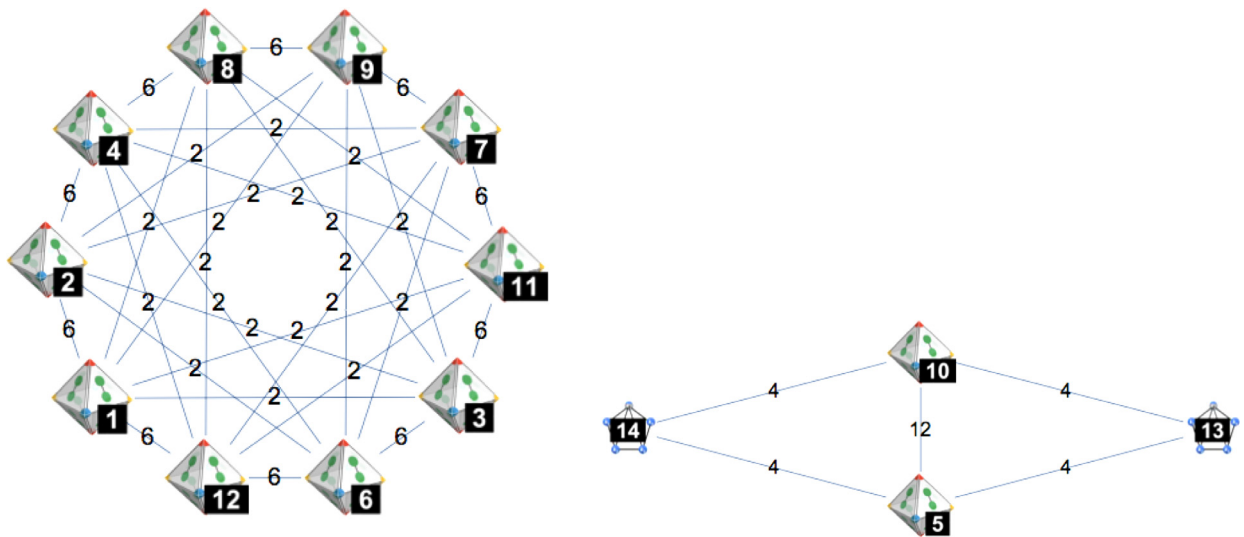


Fig. 2. Colour online. Super-cluster \mathcal{S}_{75} for $N = 3 \cdot 5^2 = 75$. Using notation from Theorem 9.7, all 10 vertices in the component $\mathcal{S}_{75}^{(1)}$ (left) have 14 wavenumbers each. In the component \mathcal{S}_{75}^* (right) the vertices numbered 5 and 10 have 12 wavenumbers each, which are strongly connected since their connecting edge has label 12. The vertices numbered 13 and 14 have 5 wavenumbers each.

again opposite faces have the same modes. The “squashed” octahedron, made out of 6 degenerate quintets (Eq. (48)), has a total of 9 different modes: 3 modes from the vertices plus $3 \times 2 = 6$ modes from its “three faces”. Finally, the two simple “pentagons”, Eq. (49), depicted in Fig. 1, top right panel, contain 5 modes each.

Now, when we consider all the clusters together it is easy to see that they must be connected. Recall that most wavenumbers in the range $1 \leq k \leq N - 1$ belong to at least one cluster, with a few exceptions depending on the divisibility of N . In the simplest case $3|N \wedge 6 \nmid N$ and a typical scenario of non-degenerate clusters, a direct counting of modes from the total number of clusters would give a total of $14 \times \lfloor \frac{N}{6} \rfloor$ modes, which is greater than $2N$, so many modes are repeated and thus some octahedron clusters are connected. This leads to the following.

Definition 9.1 (Super-cluster). Let N be the number of particles of the FPUT system, with $N > 6$ and $3|N$. A super-cluster \mathcal{S}_N is defined as an undirected graph whose vertices are either octahedron clusters, doubled octahedron clusters or pentagons, as appropriate, and where an edge connecting two vertices means that the clusters corresponding to the connected vertices have one or more common modes. We label the edge with a **cardinal** representing the number of common modes. Also, to identify the vertices properly, we label them with an **ordinal** representing their position in the succession appearing on the last lines of Eqs. (46)–(49).

Example 9.2. Consider the case $N = 3 \cdot 5^2 = 75$. This corresponds to the case $3|N \wedge 6 \nmid N$, along with $5|N$, in Table 1. The super-cluster \mathcal{S}_{75} has a total of 14 vertices, corresponding to the 12 octagon clusters from Eq. (46), where $n = 2$ corresponds to the 1-st cluster (hence labeled with the ordinal 1 in the figure), and $n = N/3 - 1 = 24$ corresponds to the 12-th cluster. The remaining 2 clusters are the ones appearing in Eq. (49). Fig. 2 shows the super-cluster \mathcal{S}_{75} , where we can identify 2 connected components (termed “component” for simplicity). By definition, the wavenumbers in a given component interact via 5-wave resonances exclusively with wavenumbers from that given component.

What is the reason for the decomposition of a super-cluster into disjoint connected components? For a given choice of N , the number of components of the super-cluster S_N can be predicted from simple arguments of divisibility. We first state a theorem that determines the common divisibility of wavenumbers inside any simple, double or squashed octahedron cluster. A second theorem states the special case of a pentagon cluster. Finally a third theorem states that a super-cluster is a disjoint union of sub-graphs such that within any sub-graph all wavenumbers are divisible by the same divisor of N .

Theorem 9.3 (Octahedra divisibility). *Let the number of particles be $N = 2^{b_0} \cdot 3^{b_1} \cdot R$, where b_0, b_1 are integers with $b_0 \geq 0$ and $b_1 \geq 1$, and R is a positive integer with $2 \nmid R \wedge 3 \nmid R$. Let r be a positive integer such that $r \mid R$.*

Consider the multi-set \mathcal{A}_n of allowed wavenumbers in a given octahedron (respectively double octahedron and “squashed” octahedron) cluster, defined as the multi-set of wavenumbers defined in Eq. (46) if $b_0 = 0$ (respectively Eq. (47) if $b_0 \geq 1$ and Eq. (48) if $b_0 \geq 2$). Then

$$r \mid k \text{ for some } k \in \mathcal{A}_n \iff r \mid k \text{ for all } k \in \mathcal{A}_n.$$

Proof. (i) Case $b_0 = 0$. Regarding the simple octahedra, by looking at Eq. (46) the allowed wavenumbers are in the multi-set $\mathcal{A}_n = \{n, N - n, n + N/3, 2N/3 - n, n + 2N/3, N/3 - n, N/3 - n/2, 2N/3 + n/2, n/2, N - n/2, N - 3n/2, 3n/2, 2N/3 - n/2, N/3 + n/2\}$ for some even integer n . So, by inspection, $r \mid n$ if and only if r divides any other number in this multi-set, because $r \mid N, N/3$ and $2N/3$. Therefore $r \mid k$ for some $k \in \mathcal{A}_n$ if and only if $r \mid k$ for all $k \in \mathcal{A}_n$.

(ii) Case $b_0 > 0$. Regarding the double octahedra, including the squashed octahedron, by looking at Eq. (47) the allowed wavenumbers are in the multi-set $\mathcal{A}_n = \{n, N - n, n + N/3, 2N/3 - n, n + 2N/3, N/3 - n, N/3 - n/2, 2N/3 + n/2, n/2, N - n/2, N - 3n/2, 3n/2, 2N/3 - n/2, N/3 + n/2, 5N/6 - n/2, N/6 + N/2, N/2 + n/2, N/2 - n/2, N/2 - 3n/2, N/2 + 3n/2, N/6 - n/2, 5N/6 + n/2\}$ for some even integer n . Again, by inspection, $r \mid n$ if and only if r divides any other number in this multi-set, because $r \mid N, N/3, 2N/3, N/6, 5N/6$ and $N/2$. Therefore $r \mid k$ for some $k \in \mathcal{A}_n$ if and only if $r \mid k$ for all $k \in \mathcal{A}_n$. \square

Remark 9.4. The restriction $2 \nmid r$ is necessary because $2 \mid n$ even if $2 \nmid \frac{n}{2}$, and the restriction $3 \nmid r$ is necessary because $3 \mid \frac{3n}{2}$ even if $3 \nmid n$.

Theorem 9.5 (Pentagon divisibility). *Under the same hypotheses of Theorem 9.3, if in addition $5 \mid R$ then one wavenumber from each pentagon quintet in Eq. (49) ($2N/3$ and $N/3$) is divisible by R . The remaining wavenumbers are divisible by $R/5$.*

Proof. The proof is direct. \square

Remark 9.6. Because the wavenumbers within a pentagon quintet have mixed divisibility properties, they will serve as connectors between clusters whose wavenumbers are divisible by R and clusters whose wavenumbers are divisible by $R/5$ only.

Theorem 9.7 (Super-cluster decomposition). *Let the number of particles be $N = 2^{b_0} \cdot 3^{b_1} \cdot R$, where b_0, b_1 are integers with $b_0 \geq 0$ and $b_1 \geq 1$, and R is a positive integer such that $2 \nmid R, 3 \nmid R$ and $R > 2^{1-b_0} \cdot 3^{1-b_1}$ (i.e. $N > 6$).*

Consider the super-cluster S_N from Definition 9.1. Then:

(i) S_N is a disjoint union of graphs over the set of divisors of R :

$$S_N = \left(\bigoplus_{\substack{d \mid R \\ d < R/5}} S_N^{(d)} \right) \bigoplus S_N^*, \quad (50)$$

where the graph $S_N^{(d)}$ is defined as the disjoint union of all connected components C of S_N such that all wavenumbers k in C satisfy $\gcd(k, R) = d$ (Eq. (53)), and the graph S_N^* is defined as the connected component of S_N such that all wavenumbers k in S_N^* satisfy $\gcd(k, R) = R$ or $\gcd(k, R) = R/5$ (if $5 \mid R$).

(ii) Provided $R \neq 1$ or $R \neq 5$, $S_N^{(d)}$ exists and is non-empty for any divisor d of R such that $d < R/5$, and the set of wavenumbers appearing in $S_N^{(d)}$ is given explicitly by

$$\mathcal{A}_N^{(d)} = \{d \cdot \ell : \ell \in \mathbb{N} \wedge 1 \leq \ell < 2^{b_0} \cdot 3^{b_1} \cdot R/d \wedge \gcd(\ell, R/d) = 1\}. \quad (51)$$

(iii) The graph S_N^* is empty if and only if $b_0 \leq 1 \wedge b_1 = 1 \wedge R \neq 1 \wedge 5 \nmid R$. When S_N^* is non-empty the set of wavenumbers appearing in S_N^* is given explicitly by

$$\mathcal{A}_N^* = \{\ell \cdot R/5 \in \mathbb{N} : \ell \in \mathbb{N} \wedge 1 \leq \ell < 2^{b_0} \cdot 3^{b_1} \cdot 5\} \setminus \mathcal{F}_N, \quad (52)$$

where the set of forbidden wavenumbers \mathcal{F}_N is defined by

$$\mathcal{F}_N = \begin{cases} \{N/6, N/3, N/2, 2N/3, 5N/6\} & \text{if } b_0 = 1 \wedge b_1 = 1, \\ \{N/3, 2N/3\} & \text{if } b_0 \neq 1 \wedge b_1 = 1, \\ \{N/2\} & \text{if } b_0 = 1 \wedge b_1 \geq 2, \\ \emptyset & \text{if } b_0 \neq 1 \wedge b_1 \geq 2. \end{cases}$$

Moreover, when $5|R$ the two pentagon quintets are vertices of S_N^* , where they serve as the only connectors between vertices whose wavenumbers k satisfy $\gcd(k, R) = R$ and vertices whose wavenumbers k satisfy $\gcd(k, R) = R/5$. Finally, when $b_0 \geq 2$ the squashed octahedron cluster is a vertex of S_N^* and is in the subgraph of S_N^* made out of vertices whose wavenumbers k satisfy $\gcd(k, R) = R$.

Proof. (i) Consider any connected component \mathcal{C} of S_N , and any wavenumber k in \mathcal{C} . Define $d_{\mathcal{C}} = \gcd(k, R)$. Assume $d_{\mathcal{C}} < R/5$, thus excluding precisely the cases $d_{\mathcal{C}} = R$ and $d_{\mathcal{C}} = R/5$, namely the only cases where the component \mathcal{C} may contain the pentagon quintets as vertices (see Theorem 9.5). It is then easy to see that all wavenumbers k' in \mathcal{C} satisfy $d_{\mathcal{C}} = \gcd(k', R)$. This is because from Theorem 9.3 all wavenumbers in the octahedron clusters (i.e., the vertices of \mathcal{C}) that contain k share the common divisor $d_{\mathcal{C}}$, but this divisor is common to all vertices in \mathcal{C} due to the fact that by definition there is a path that connects any two vertices in \mathcal{C} via common-mode connections. So, in summary we have a way to classify a connected component \mathcal{C} of S_N by the divisor

$$d_{\mathcal{C}} = \gcd(k, R) \quad \text{for all } k \in \mathcal{C}.$$

To account for the possibility that several connected components of S_N have the same divisor $d_{\mathcal{C}}$, we define

$$S_N^{(d)} = \bigoplus_{\substack{\mathcal{C} \subseteq S_N \\ d_{\mathcal{C}} = d}} \mathcal{C}, \quad \text{where } d|R \text{ and } d < R/5. \quad (53)$$

Trivially, we have $S_N^{(d)} \cap S_N^{(d')} = \emptyset$ if $d \neq d'$, because the inequality $\gcd(k, R) \neq \gcd(k, R)$ cannot be satisfied for any wavenumber k . Thus we get preliminarily $S_N = \left(\bigoplus_{\substack{d|R \\ d < R/5}} S_N^{(d)} \right) \oplus S_N^*$, where S_N^* is the remaining graph (to be determined in part (iii))

whose vertices contain wavenumbers that are divisible by $R/5$ (if $5|R$) and R .

(ii) When $d|R$ but $d < R/5$ (the condition $R \neq 1$ or $R \neq 5$ is necessary so that such a d exists), the set $\mathcal{A}_N^{(d)}$ of wavenumbers in $S_N^{(d)}$ is equal to the set of integers k between 1 and $N-1$ such that $\gcd(k, R) = d$. This is because, as demonstrated in Section 8, the set of all wavenumbers contained in the octahedron clusters include all integers between 1 and $N-1$, except perhaps for the wavenumbers $k = N/6, N/3, N/2, 2N/3, 5N/6$ (when they exist, depending on the case), which satisfy $\gcd(k, R) = R$ so these exceptions do not apply when $d < R/5$. Now, explicitly:

$$\mathcal{A}_N^{(d)} = \{k \in \mathbb{N} : 1 \leq k < N \wedge \gcd(k, R) = d\} = \left\{ d \cdot \ell : \ell \in \mathbb{N} \wedge 1 \leq \ell < \frac{N}{d} \wedge \gcd\left(\ell, \frac{R}{d}\right) = 1 \right\}.$$

(iii) By definition, the set \mathcal{A}_N^* of wavenumbers in S_N^* is the union of the set of wavenumbers divisible by R and the set of wavenumbers divisible by $R/5$, minus the set of non-interacting wavenumbers discussed in Section 8. Explicitly:

$$\mathcal{A}_N^* = (\mathcal{A}_N^{(R/5)} \cup \mathcal{A}_N^{(R)}) \setminus \mathcal{F}_N,$$

which simplifies to Eq. (52).

(\Leftarrow) If $b_0 \leq 1$ and $b_1 = 1$ then the set of wavenumbers divisible by R , $\mathcal{A}_N^{(R)} = \{k \in \mathbb{N} : 1 \leq k < N \wedge \gcd(k, R) = R\}$, contains exactly the forbidden wavenumbers, i.e. the non-interacting wavenumbers discussed in Section 8 under Cases 1 and 2. The condition $R \neq 1$ is kept for bookkeeping purposes only since it follows from the hypothesis $R > 2^{1-b_0} \cdot 3^{1-b_1}$, which implies $R = 2^{1-b_0} > 1$. If $5 \nmid R$ then the set $\mathcal{A}_N^{(R/5)}$ of wavenumbers divisible by $R/5$ is empty (in fact, it is not defined). Therefore

$$S_N^* = \emptyset \quad \text{if } b_0 \leq 1 \text{ and } b_1 = 1 \text{ and } R \neq 1 \text{ and } 5 \nmid N.$$

(\Rightarrow) For any other condition on b_0 and b_1 it is easy to see that the set of wavenumbers divisible by R minus the set of non-interacting wavenumbers is non-empty by just calculating the set $\{k \in \mathbb{N} : 1 \leq k < N \wedge \gcd(k, R) = R\} = \{R, 2R, \dots, (2^{b_0} \cdot 3^{b_1} - 1)R\}$ and subtracting from it the set of forbidden wavenumbers $\mathcal{F}_N \subseteq \{N/6, N/3, N/2, 2N/3, 5N/6\}$. When $b_0 \geq 2$, it follows $12|N$ so the squashed octahedron, Eq. (48), exists and all its wavenumbers are divisible by R , so the squashed octahedron is a vertex of S_N^* . When $5|N$ then S_N^* contains the two pentagon quintets so it is not empty. When $R = 1$ the hypothesis $N > 6$ implies either $b_0 \geq 2 \wedge b_1 = 1$ or $b_0 \geq 0 \wedge b_1 \geq 2$, which were covered at the beginning of this paragraph. Finally, when they exist, the pentagon clusters are the only vertices containing wavenumbers that have different common divisors with R : $R/5$ and R . Thus these vertices are the only possible connectors between subgraphs containing wavenumbers that are divisible by R and subgraphs containing wavenumbers that are divisible by $R/5$ only. \square

Remark 9.8. The size of the sets of connected wavenumbers $\mathcal{A}_N^{(d)}$ and \mathcal{A}_N^* , defined in points (ii) and (iii) above, can be calculated in terms of Euler's totient function $\phi(n)$, to be defined in Section 11, Eq. (54). The results are:

$$\# \mathcal{A}_N^{(d)} = 2^{b_0} \cdot 3^{b_1} \phi(R/d), \quad \# \mathcal{A}_N^* = -\# \mathcal{F}_N + \begin{cases} 2^{b_0} \cdot 3^{b_1} \cdot 5 - 1 & \text{if } 5|R, \\ 2^{b_0} \cdot 3^{b_1} - 1 & \text{if } 5 \nmid R, \end{cases}$$

where

$$\# \mathcal{F}_N = \begin{cases} 5 & \text{if } b_0 = 1 \wedge b_1 = 1, \\ 2 & \text{if } b_0 \neq 1 \wedge b_1 = 1, \\ 1 & \text{if } b_0 = 1 \wedge b_1 \geq 2, \\ 0 & \text{if } b_0 \neq 1 \wedge b_1 \geq 2. \end{cases}$$

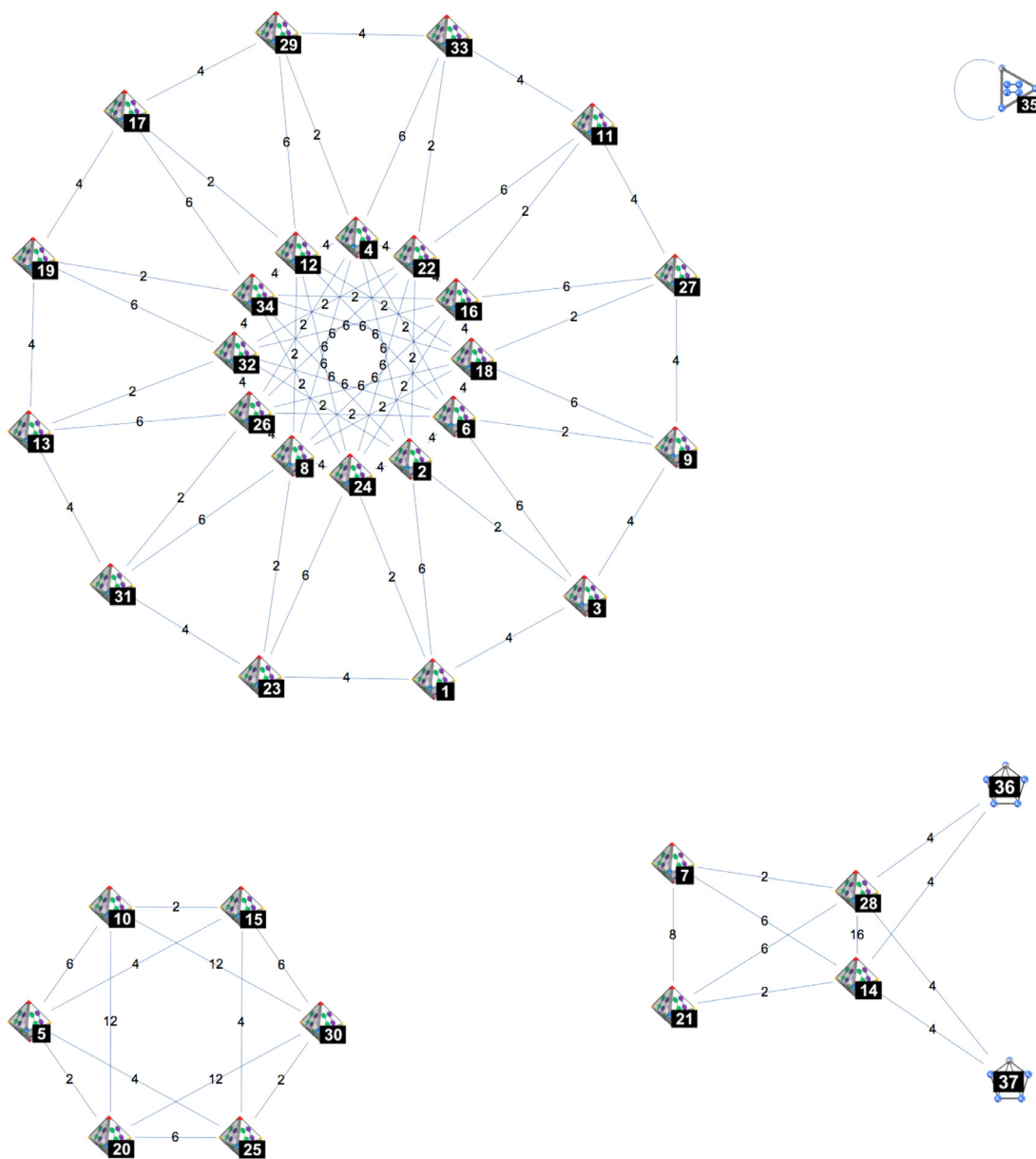


Fig. 3. Colour online. Super-cluster S_{420} for $N = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. All 24 vertices in the component $S_{420}^{(1)}$ (top left) and all 6 vertices in the component $S_{420}^{(5)}$ (bottom left) have 22 wavenumbers each. The component S_{420}^* is the disjoint union of the two subgraphs at the right of the figure. In this component, the vertices labeled 7 and 21 have 22 wavenumbers each, and the vertices labeled 28 and 14 have 20 wavenumbers each, which are strongly connected since their connecting edge has label 16. They are connected to the pentagon quintets, (vertex labels 36 and 37) which have 5 wavenumbers each. The isolated vertex with label 35 is the squashed octahedron.

Example 9.9. Consider the case $N = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$, so $b_0 = 2$, $b_1 = 1$ and $R = 5 \cdot 7$. The set of divisors of R that are less than $R/5$ is $\{1, 5\}$. From Eq. (50) the super-cluster has the following decomposition:

$$S_N = S_N^{(1)} \oplus S_N^{(5)} \oplus S_N^*,$$

where S_N^* , containing by definition the wavenumbers that are divisible by $R (= 35)$ or $R/5 (= 7)$, is a disjoint union of two graphs (see Fig. 3, top and bottom right graphs). Also in the figure, we see graph $S_N^{(1)}$ (top left), made out of wavenumbers that are not divisible by 5 nor by 7, and graph $S_N^{(5)}$ (bottom left), made out of wavenumbers that are divisible by 5 but not by 7. Vertices are mainly double octahedron clusters but we can observe the two pentagon quintets and the squashed octahedron cluster. As for the set of wavenumbers in each component of the supercluster, we have, using Remark 9.8: The set $\mathcal{A}_N^{(1)}$ of wavenumbers in $S_N^{(1)}$, has a total of $\#\mathcal{A}_N^{(1)} = 2^{b_0} \cdot 3^{b_1} \phi(R) = 288$ wavenumbers. The set $\mathcal{A}_N^{(5)}$ of wavenumbers

Table 2

Study of lowest order of FPUT irreducible resonances that are not of Birkhoff normal form (in other words, resonances that effectively exchange energy amongst modes), as a function of N . The cases admitting 5-wave resonances are highlighted in **boldface**. 4-wave resonances are always in so-called resonant Birkhoff normal form, which do not produce effective energy transfers throughout the whole spectrum of modes. In contrast, 5- and 6-wave irreducible resonances cannot be simplified in terms of resonant Birkhoff normal forms, because they mix energies over a wide range of modes.

$N \leq 6$	Lowest-order Resonances	Type of Resonance
3	6-wave	Pairing-off
4	4-wave & 6-wave	Pairing-off
5	6-wave	Pairing-off
6	4-wave & 6-wave	Pairing-off & Cyclotomic
$N > 6$	Lowest-order Resonances	Type of Resonance
1 (mod 6) (7, 13, 19, ...)	6-wave	Pairing-off
2 (mod 6) (8, 14, 20, ...)	4-wave & 6-wave	Pairing-off
3 (mod 6) (9, 15, 21, ...)	5-wave	Cyclotomic
4 (mod 6) (10, 16, 22, ...)	4-wave & 6-wave	Pairing-off
5 (mod 6) (11, 17, 23, ...)	6-wave	Pairing-off
0 (mod 6) (12, 18, 24, ...)	4-wave & 5-wave	Pairing-off & Cyclotomic

in $S_N^{(5)}$, has a total of $\#A_N^{(5)} = 2^{b_0} \cdot 3^{b_1} \phi(R/5) = 72$ wavenumbers. The set A_N^* of wavenumbers in S_N^* , has a total of $\#A_N^* = -2 + 2^{b_0} \cdot 3^{b_1} \cdot 5 - 1 = 57$ wavenumbers. They total 417 wavenumbers, namely two less than the universe of wavenumbers $(N-1)$ due to the fact that there are two forbidden wavenumbers: $\{N/3, 2N/3\}$.

10. Route to thermalisation: Combined analysis of 4-, 5- and 6-wave resonances as a function of the number of particles N

The results of the previous Sections can be used to study how the lowest possible order of the resonances depends very sensitively on the number of interacting particles N . For example, as is well known, 4-wave resonance solutions that are non-trivial (i.e. those that do not contribute to a Birkhoff normal form in the Hamiltonian, or in our notation those that are not composite resonances) exist for N even only. These non-trivial 4-wave resonances admit a so-called *resonant Birkhoff normal form* [5]. Thus, if N is odd, the lowest non-trivial order of a resonance is either 5 or 6. As we found in the last Sections, 5-wave resonances exist if and only if $3|N$ and $N > 6$. Thus, for N odd, if $N \leq 6$ or $3 \nmid N$ then the lowest non-trivial resonant order is 6. Table 2 summarises the results as a function of N . In this table we always include the next order when 4-wave resonances exist (i.e. for N even) because, as is well known, if one truncates a system up to the 4-wave resonances the result is an integrable system (resonant Birkhoff normal form) due to the fact that the resonant quartets, whose wavenumbers are defined in Eq. (31), are not connected: the wavenumbers in a given quartet do not belong to any other quartets. Thus, the next order of resonance must be included in order to obtain complex (and typically chaotic) dynamics characterised by energy mixing across all wavenumbers, which will lead to an understanding of the thermalisation process. We remark that our study produces irreducible 5- and 6-wave resonances, which by definition cannot be simplified in terms of 4-wave resonant Birkhoff normal forms.

Super-clusters modify the thermalisation hypothesis. Writing $N = 2^{b_0} \cdot 3^{b_1} \cdot R$ with $2 \nmid R$, $3 \nmid R$, $b_0 \geq 0$ and $b_1 \geq 1$, Theorem 9.7 establishes that the set of divisors d of R separates the universe of modes (with wavenumbers $1, \dots, N-1$) into disjoint sets $S_N^{(d)}$ (including S_N^*) of interacting modes via 5-wave resonances. Roughly speaking, wavenumbers in a given set must be divisible by d but are not divisible by any $d' > d$. Physically, this result modifies the thermalisation hypothesis: modes whose wavenumbers belong to disjoint sets do not interact via 5-wave resonances, but thermalisation is possible within each disjoint set $S_N^{(d)}$.

What about 4-wave resonances in combination with 5-wave resonances? 4-wave resonances that exchange energy across modes exists when N is even only, or in our notation when $b_0 \geq 1$. Hence for N odd the situation is as described in the previous paragraph. For N even, the wavenumbers involved in a resonant quartet are shown in Eq. (31): $\{k_1, N/2 - k_1, N - k_1, N/2 + k_1\}$, where $k_1 = 1, \dots, [N/4]$. By inspection, given k_1 all four wavenumbers share the same divisors d with R . Therefore energy-exchanging 4-wave resonances occur within the same disjoint set $S_N^{(d)}$ of the super-cluster. So, in terms of energy exchanges, for N even the state space still separates as a direct sum of lower-dimensional sub-spaces, associated with each disjoint set $S_N^{(d)}$. On each of these sub-spaces thermalisation may hold independently, but this is a matter to be studied further both numerically and analytically in a subsequent paper.

It is worth mentioning that “trivial” or composite 4-wave resonances (e.g. $k_1 + k_2 = k_1 + k_2$) exist for any N and they do allow interactions across disjoint components of the super-clusters. However these interactions are subtle: they do not exchange energy but they mix up the Fourier phases across disjoint components. So, in terms of energy exchanges, the picture of independent ergodic components $S_N^{(d)}$ still applies. In particular, energy must be conserved separately on each component.

Therefore, in order to obtain global mixing we need to go to higher orders: pairing-off 6-wave resonances will mix up the energies across disjoint components $\mathcal{S}_N^{(d)}$ of the super-cluster. To see this, consider the simple case $b_0 = 0, b_1 \geq 1$, so N is odd and thus pairing-off sextuplet solutions are given in Eq. (32): wavenumbers k_1, k_2, k_3 and their conjugate pairs $N - k_j$ interact if and only if $k_1 + k_2 + k_3 = N$. Therefore, while a wavenumber k_j and its conjugate pair $N - k_j$ share the same greatest common divisor d with R (i.e. they belong to the same disjoint component $\mathcal{S}_N^{(d)}$ of the super-cluster), it is not necessary that the three wavenumbers k_1, k_2, k_3 share the same greatest common divisor with R . For example, we could take wavenumber k_1 from component $\mathcal{S}_N^{(d_1)}$ and wavenumber k_2 from component $\mathcal{S}_N^{(d_2)}$, where $d_2 \neq d_1$ so $\mathcal{S}_N^{(d_1)} \cap \mathcal{S}_N^{(d_2)} = \emptyset$. The third wavenumber, $k_3 = N - k_1 - k_2$, will belong to some component $\mathcal{S}_N^{(d_3)}$, where d_3 is restricted by the condition $\gcd(d_1, d_2) | d_3$. If $\gcd(d_1, d_2) = 1$ then this restriction is trivial. For example, consider $N = 3 \cdot 5 \cdot 7 \cdot 13$ and take $k_1 = 7 \in \mathcal{S}_N^{(7)}$, $k_2 = 13 \in \mathcal{S}_N^{(13)}$ and thus $k_3 = N - 7 - 13 = N - 20 \in \mathcal{S}_N^{(5)}$.

In conclusion, to achieve full-scale mixing and thermalisation it is necessary to go to 6-wave resonant interactions. If $3|N$ and $N > 6$, then 5-wave resonant interactions may provide partial mixing and thermalisation within separated regions of wavenumber space, the largest one being $\mathcal{S}_N^{(1)}$ (or \mathcal{S}_N^* if $R = 1$ or $R = 5$), defined as the component of the super-cluster whose wavenumbers k do not have any common factors with N , except for the factors 1, 2, 3 (and 5, if $R = 5$) or products of powers thereof.

The 5-wave resonances in the large- N limit. To see how results could depend sensitively on the number of particles, notice that all numerical studies we are aware of have been performed for the cases $N = 2^b$, i.e. N is a power of 2. In such a case the only possible resonances are of pairing-off form, as demonstrated in Corollary 11.5 and explicitly constructed in Section 4. In such a case, 4-wave and more importantly 6-wave resonances will produce significant energy mixing across scales (and thus thermalisation) because the resonant sextuplets, Eqs. (33)–(35), have 2 free parameters so there are $\mathcal{O}(N^2)$ sextuplets, covering uniformly and surjectively the universe of available wavenumbers, of size $\mathcal{O}(N)$; therefore there will be a huge number of connections across resonant sextuplets, much like in the well-known case of triad interactions in the Euler fluid equations. The time scale of the thermalisation process is determined by the order of the resonance: 6-wave resonant interactions in this case.

However, changing the number of particles by just one unit can completely change the picture. Writing $N = 2^b - (-1)^b$ for integer $b \geq 4$, we can show $2 \nmid N$, $3|N$ and $N \geq 15$, so 5-wave interactions would take place which could generate partial thermalisation within connected components of the super-cluster. These interactions would be a bit faster than 6-wave interactions so they could dominate the dynamics for quite a while before 6-wave interactions take over. It will be interesting to compare these near-power-of-two cases as the integer b runs over a large interval: $b \in [4, 200]$, corresponding to N ranging from 15 to about $1.6 \cdot 10^{60}$. Writing as usual $N = 3^{b_1} \cdot R$ where $2 \nmid R$ and $3 \nmid R$, the quantities to study are:

- (i) The total number of components in the super-cluster, $D(\mathcal{S}_N) \equiv \#\{d \in \mathbb{N} : d|R \wedge d < R/5\} (+1 \text{ if } \mathcal{S}_N^* \text{ is non-empty})$, which is roughly equal to the total number of divisors of R , denoted in the literature as $\sigma_0(R)$ or $d(R)$. In attention to the fact that $R/5$ and R are gathered into one component \mathcal{S}_N^* , and that in some instances \mathcal{S}_N^* is empty, the total number of components in the super-cluster may differ from $\sigma_0(R)$ by one unit in some instances.
- (ii) The total amount of wavenumbers that belong to $\mathcal{S}_N^{(1)}$, the largest component of the super-cluster. This is equal to the size of the set $\mathcal{A}_N^{(1)}$ as defined in Eq. (51). In terms of R this is equal to

$$\#\mathcal{A}_N^{(1)} = \#\{\ell \in \mathbb{N} : 1 \leq \ell < 3^{b_1} \cdot R \wedge \gcd(\ell, R) = 1\} = 3^{b_1} \cdot \phi(R),$$

where $\phi(n)$ is Euler's totient function, defined below in Eq. (54).

Fig. 4 (left panel) plots in blue open circles the number of disjoint components of the super-cluster, $D(\mathcal{S}_N)$, for near-power-of-two values of N ranging from 15 to 1.6×10^{60} . The function oscillates wildly between a lower bound of 1 component (corresponding to the case $N = 3^{b_1} \cdot p$ with p an odd prime) and an upper bound that grows as N grows, reaching up to $\approx 1.57 \times 10^6$ at $N = 2^{180} - 1 \approx 1.5 \times 10^{54}$. In the same plot, an upper bound is shown (red solid line), representing Wigert's [23] maximal order of $\sigma_0(R)$, namely $\ln(2)\ln(R)/\ln(\ln(R))$, which satisfies

$$\limsup_{n \rightarrow \infty} \frac{\sigma_0(n)}{\ln(2)\ln(n)/\ln(\ln(n))} = 1.$$

As evidenced in the figure, this bound on $\sigma_0(R)$ is nearly saturated only a couple of times at early values of N (6.7×10^{10} and 1.15×10^{18}).

Fig. 4 (right panel) plots the relative size of the largest component of the super-cluster, for near-power-of-two values of N ranging from 15 to 1.6×10^{60} . First of all, for many values of N , and uniformly over the studied range, this ratio is close to 1, meaning this single super-cluster component $\mathcal{S}_N^{(1)}$ dominates over all the other components. This is reinforced by Fig. 4 (bottom panel), which shows that cases with ratio = 1 occur with high probability. Second, there is an evident spread of the ratio, from 1 down to 0.473 at $N = 2^{180} - 1 \approx 1.5 \times 10^{54}$. This point corresponds also to the largest instance of the number of components $D(\mathcal{S}_N)$, which is not a coincidence as Fig. 4 (bottom panel) demonstrates a significant correlation between low ratios and large number of components.

In conclusion, the large- N limit of the FPUT system in the regimes dominated by 5-wave resonances will have at most a probabilistic interpretation, in the sense that only averages over a given set of values of N make sense. Any result will depend sensitively on the exact choice of the value of N . Even the way we take the limit will matter: the case $N = 2^b$, $b \rightarrow \infty$

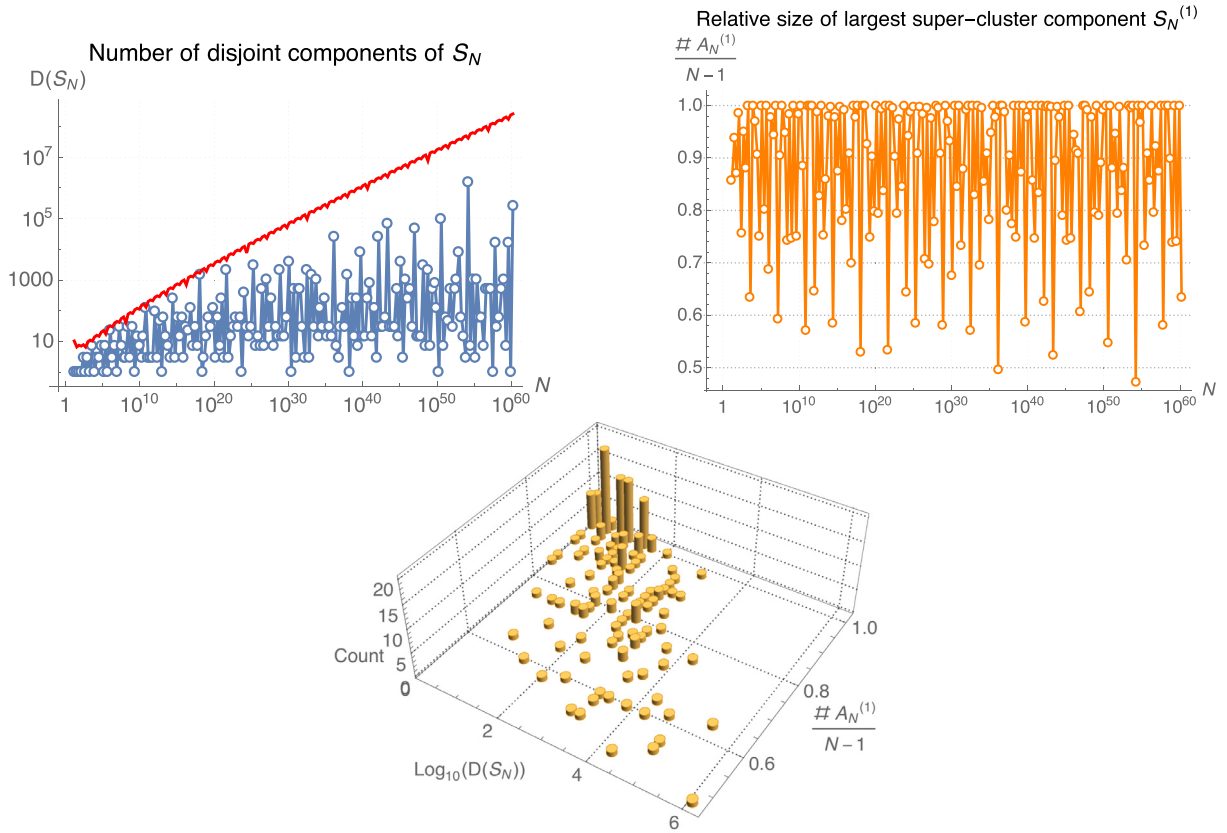


Fig. 4. Colour online. Left panel: Log-log plot of $D(S_N)$, the total number of disjoint components of the super-cluster S_N , as a function of the number of particles N of the form $N = 2^b - (-1)^b$, for $b \in [4, 200]$ (blue solid lines and open circles). The upper bound (red solid line) is Wigert's bound [23] which in our notation reads $D(S_N) \approx \exp\left(\frac{\ln(2) \ln(R)}{\ln(\ln(R))}\right)$, where $N = 3^{b_1} \cdot R$. Right panel: Log-linear plot of $\frac{\# A_N^{(1)}}{N-1}$, namely the proportion of wavenumbers in the largest super-cluster component $S_N^{(1)}$ (with respect to the total of available modes, $N-1$), as a function of the number of particles N of the form $N = 2^b - (-1)^b$, for $b \in [4, 200]$. Bottom panel: Combined 3D histogram of the two quantities $D(S_N)$ and $\frac{\# A_N^{(1)}}{N-1}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

is dominated by 6-wave resonances and the case $N = 2^b \cdot 3$, $b \rightarrow \infty$ is dominated by 5-wave resonances with a deceptively simple super-cluster given by just one component: $S_N = S_N^*$, because $R = 1$.

The thermodynamic limit. It is worth mentioning that in the thermodynamic limit, i.e. $N \rightarrow \infty$, $L \rightarrow \infty$, with L the length of the chain, keeping constant the linear density of masses, 4-wave resonant interactions become important for thermalisation, see [13,14]. This is because the Fourier space becomes dense and the re-scaled wavenumbers k/N become real in the interval $(0, 1)$, allowing for a continuous resonant manifold.

However, note also that in the thermodynamic limit the effective nonlinear resonance broadening may allow near-resonant 3-wave interactions to be more effective than resonant 4-wave interactions. The interplay between near-resonant 3-wave interactions and 4-wave interactions will be the subject of our future research.

11. Mathematical underpinnings of FPUT resonances: A basis for the real FPUT polynomials

In this Section we provide a rigorous mathematical construction of the general solution regarding M -wave resonances in the FPUT system. The main results are Theorems 11.4 and 11.10, along with Corollaries 11.7 and 11.11, which establish a basis for the real FPUT polynomials defined in Section 6. It is worth mentioning Corollary 11.5, which shows that if N is a prime or a power of 2, then the only possible resonances are of pairing-off form (discussed in Section 4).

We begin with two definitions that will be essential in what follows. First, we recall Definition 7.1 of Cyclotomic polynomial $\Phi_n(x)$. Second, the following

Definition 11.1 (Euler's totient function). The degree of the cyclotomic polynomial $\Phi_n(x)$ is known as Euler's totient function $\phi(n)$, defined by

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad (54)$$

where the product is over all primes p that divide n . Notice that $\phi(n) \leq \frac{n}{2}$ when n is even. If n is a power of 2, then $\phi(n) = \frac{n}{2}$. Also, $\phi(n)$ is even if $n \geq 3$.

Although there is no general formula to obtain the cyclotomic polynomial $\Phi_n(x)$ for arbitrary $n \in \mathbb{N}$, it is possible to construct it algorithmically in a very fast way. For example, *Mathematica* software as has a built-in function `Cyclotomic[n,x]` which produces $\Phi_n(x)$.

For a given number of particles N , define as usual $\zeta = \exp(i\pi/N)$. Define also $\phi = \phi(2N)$, the Euler's totient function of $2N$. The basis for the real FPUT polynomials $\rho(x)$ has dimension $N - \phi/2 - 1$ and consists of two parts:

1. The basis \mathcal{K}_N of the kernel of the map $\rho(x) \rightarrow \rho(\zeta)$, with dimension $N - \phi$.
2. Any complementary basis \mathcal{P}_N , made out of pairing-off binomials, with dimension $\phi/2 - 1$. For example we can choose

$$\mathcal{P}_N = \{x - x^{N-1}, \dots, x^{\phi/2-1} - x^{N-\phi/2+1}\}.$$

11.1. Integer basis for real polynomials

As above, $\zeta \in \mathbb{C}$ denotes a primitive $2N$ th root of unity.

We wish to construct or enumerate all real polynomials; i.e. all integer polynomials

$$\rho(x) = \sum_{k=1}^{N-1} \rho_k x^k$$

which evaluate to real numbers when ζ is substituted for x .

Let P denote the additive group of all polynomials in x of degree at most $N - 1$ with integer coefficients. By definition, every element of P has a unique expression of the form $\sum_{k=0}^{N-1} \rho_k x^k$ with $\rho_k \in \mathbb{Z}$; we say that $\{1, x, \dots, x^{N-1}\}$ is a \mathbb{Z} -basis of P . (Since it has a \mathbb{Z} -basis with N elements, P is said to be a (free abelian) group of rank N .)

Let C denote the additive subgroup of \mathbb{C} consisting of all polynomials with integer coefficients of any degree, in ζ .

The following (well-known) theorem establishes the dimension and appropriate bases of the group C and its subgroup of real elements.

Theorem 11.2. (i) $\zeta^r \in C$ for all $r \in \mathbb{Z}$. Furthermore, for any $r \in \mathbb{Z}$ the set

$$\{\zeta^r, \zeta^{r+1}, \dots, \zeta^{r+\phi-1}\}$$

is a \mathbb{Z} -basis of C .

(ii) Let $C^+ := C \cap \mathbb{R}$. Then C^+ is a subgroup of C of rank $\phi/2$. The set

$$\{1, \zeta + \zeta^{-1}, \dots, \zeta^{\phi/2-1} + \zeta^{1-\phi/2}\}$$

is a \mathbb{Z} -basis of C^+ .

Proof. (i) First note that since $\zeta^{2N} = 1$, we have $\zeta^r = \zeta^{r+2tN}$ for any t and thus $\zeta^r \in C$ even for negative r .

It follows that multiplication by ζ^r is an isomorphism from C to itself (whose inverse is multiplication by ζ^{-r}). So it is enough to show that $1, \zeta, \dots, \zeta^{\phi-1}$ is a \mathbb{Z} -basis of C .

Now $\Phi_{2N}(x) \in P$ is an integer polynomial of degree ϕ with leading term x^ϕ . Since $\Phi_{2N}(\zeta) = 0$, we obtain a relation $\zeta^\phi = \sum_{i=0}^{\phi-1} a_i \zeta^i$ so that ζ^ϕ lies in the integer span of $\{\zeta^0, \zeta^1, \dots, \zeta^{\phi-1}\}$. By induction, suppose that ζ^t , $t \geq \phi$, lies in this span. Then $\zeta^{t+1} = \zeta \cdot \zeta^t$ lies in the span of $\{\zeta^1, \dots, \zeta^\phi\}$ and hence also in the span of $\{\zeta^0, \zeta^1, \dots, \zeta^{\phi-1}\}$.

Finally, suppose that $\sum_{i=0}^{\phi-1} a_i \zeta^i = 0$. Then $a_i = 0$ for all i , since ζ is not a root of any nonzero rational polynomial of degree less than ϕ .

(ii) Since $\zeta^{1-\phi/2}, \zeta^{2-\phi/2}, \dots, 1, \zeta, \dots, \zeta^{\phi/2-1}, \zeta^{\phi/2}$ is a \mathbb{Z} -basis of C by part 1, it follows that the real numbers

$$1, \zeta + \zeta^{-1}, \dots, \zeta^{\phi/2-1} + \zeta^{1-\phi/2}$$

are linearly independent over \mathbb{Z} . So the span of these elements is contained in C^+ and is linearly independent. We need to show that this span is all of C^+ :

Suppose that

$$\sigma = a_{1-\phi/2} \zeta^{1-\phi/2} + \dots + a_0 + a_1 \zeta + \dots + a_{\phi/2-1} \zeta^{\phi/2-1} + a_{\phi/2} \zeta^{\phi/2} \in C$$

is real; i.e. suppose that $\sigma \in C^+$. Then, since $\bar{\zeta}^i = \zeta^{-i}$, we have

$$\sigma = \bar{\sigma} = a_{\phi/2} \zeta^{-\phi/2} + a_{\phi/2-1} \zeta^{1-\phi/2} + \dots + a_{1-\phi/2} \zeta^{\phi/2-1}$$

and hence, comparing terms, we get

$$a_i = a_{-i} \text{ for } 0 \leq i \leq \phi/2, \text{ and } a_{\phi/2} = 0$$

by the linear independence of part 1.

It follows that $\sigma = a_0 + \sum_{i=1}^{\phi/2-1} a_i(\zeta^i + \zeta^{-i})$ lies in the span of

$$\{1, \zeta + \zeta^{-1}, \dots, \zeta^{\phi/2-1} + \zeta^{1-\phi/2}\}$$

as required. \square

From the fact that $\zeta^N = -1$ we can rewrite this last \mathbb{Z} -basis of C^+ in terms of positive powers of ζ :

$$B_+ = \{1, \zeta - \zeta^{N-1}, \dots, \zeta^{\phi/2-1} - \zeta^{N-\phi/2+1}\}. \quad (55)$$

Theorem 11.2 implies that the individual powers of ζ appearing in the above basis, along with the element $\zeta^{N-\phi/2}$, form a ϕ -dimensional \mathbb{Z} -basis of C :

$$B = \{1, \zeta, \dots, \zeta^{\phi/2-1}, \zeta^{N-\phi/2}, \zeta^{N-\phi/2+1}, \dots, \zeta^{N-1}\}.$$

From the fact that the elements of B are linearly independent it follows that a polynomial supported by the basis B is real if and only if it is supported by the basis B_+ . In particular, the coefficient of $\zeta^{N-\phi/2}$ of a real polynomial supported by the basis B is identically zero:

Corollary 11.3. *The expression*

$$a_0 + \sum_{k=1}^{\phi/2-1} a_k \zeta^k + \sum_{k=N-\phi/2}^{N-1} a_k \zeta^k \in C$$

is real if and only if $a_k = a_{N-k}$ for $k = 1, \dots, \phi/2 - 1$ and $a_{N-\phi/2} = 0$: i.e. if and only if the terms pair off (see definition of pairing off in Section 4).

11.2. Finding real polynomials

11.2.1. Trivially real polynomials and pairing-off resonances

Let \mathcal{R} denote the additive subgroup of P consisting of real polynomials with integer coefficients; i.e. the group of polynomials $\rho(x)$ such that $\rho(\zeta) \in \mathbb{R}$.

Our interest is in calculating the real polynomials which do not arise from pairing off terms. The key to doing this is to understand the kernel of the evaluation map

$$P \rightarrow C, \quad \alpha(x) \mapsto \alpha(\zeta).$$

We denote this kernel by \mathcal{K} .

This evaluation map is a homomorphism of groups (even of rings). By Theorem 11.2 it is a surjective map. It follows that its kernel, \mathcal{K} , is a free abelian group of rank $N - \phi$.

We thus have:

Theorem 11.4. *Let B be any \mathbb{Z} -basis of \mathcal{K} . Then*

$$B \cup \{1, x - x^{N-1}, \dots, x^{\phi/2-1} - x^{N-\phi/2+1}\}$$

is a \mathbb{Z} -basis of the group of real polynomials \mathcal{R} .

In particular, the group of real polynomials has rank $N - \phi/2$.

Proof. Suppose that $\alpha(x) \in \mathcal{R}$. Then $\alpha(\zeta) \in \mathbb{R}$. By Theorem 11.2 (ii), there exist unique $\lambda_k \in \mathbb{Z}$ such that

$$\alpha(\zeta) = \lambda_0 + \sum_{k=1}^{\phi/2-1} \lambda_k (\zeta^k - \zeta^{N-k}).$$

Let $\beta(x) := \lambda_0 + \sum_{k=1}^{\phi/2-1} \lambda_k (x^k - x^{N-k})$. Then $\alpha(x) - \beta(x) \in \mathcal{K}$. Hence there exist $\mu_p \in \mathbb{Z}$ such that

$$\alpha(x) - \beta(x) = \sum_{p(x) \in B} \mu_p p(x).$$

Thus $\alpha(x) = \sum_{p(x) \in B} \mu_p p(x) + \lambda_0 + \sum_{k=1}^{\phi/2-1} \lambda_k (x^k - x^{N-k})$. \square

On the other hand, let us say that $\alpha(x) \in \mathcal{R}$ is *trivially real* if it is of the form

$$\alpha(x) = \lambda_0 + \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \lambda_k (x^k - x^{N-k}).$$

Thus a trivially real polynomial is precisely one which gives rise to a *pairing-off resonance* (see Section 4). We observe that the group of trivially real polynomials has basis

$$\{1, x - x^{N-1}, \dots, x^{\lfloor \frac{N-1}{2} \rfloor} - x^{N-\lfloor \frac{N-1}{2} \rfloor}\}.$$

In particular this is a free abelian group of rank $\lfloor \frac{N-1}{2} \rfloor + 1 = \lceil N/2 \rceil$.

Corollary 11.5. Every real polynomial is trivially real if and only if

$$\phi/2 = N - \lceil N/2 \rceil.$$

This happens if and only if either N is a power of 2 or an odd prime number p .

Proof. We begin by noting that $\phi/2 \leq N - \lceil N/2 \rceil$ with equality if and only if either N is a power of 2 or an odd prime number p .

For if $N = 2^a p_1^{b_1} \cdots p_t^{b_t}$ then $\phi = \phi(2N) = N \cdot (1 - 1/p_1) \cdots (1 - 1/p_t)$. Thus $\phi \leq N$ with equality only if $t = 0$; i.e. N is a power of 2.

When N is even, we have $N - \lceil N/2 \rceil = N/2$ and $\phi/2 \leq N/2$ with equality if and only if N is a power of 2.

When N is odd, we have $N - \lceil N/2 \rceil = (N - 1)/2$ and $\phi/2 \leq (N - 1)/2$ with equality if and only if $\phi = N - 1$; i.e. if and only if $N = p_1$ is prime.

Let $\mathcal{T} \subset \mathcal{R}$ be the group of trivially real polynomials. We have $\text{rank}(\mathcal{T}) = \lceil N/2 \rceil$ and $\text{rank}(\mathcal{R}) = N - \phi/2$, so that $\mathcal{T} \subset \mathcal{R}$ with equality only if $N = 2^b$ or $N = p$.

Observe that if $N = 2^b$, then $\phi = N$ so that $\mathcal{K} = \{0\}$ and $\mathcal{T} = \mathcal{R}$ by the proof of Theorem 11.4, while if $N = p$ then $N - \phi = 1$ and $\{\Phi_{2p}(x)\}$ is a basis of \mathcal{K} . However

$$\Phi_{2p}(x) = 1 - x + x^2 - \cdots - x^{p-1} = 1 - (x - x^{p-1}) + \cdots \pm (x^{(p-1)/2} - x^{(p+1)/2}) \in \mathcal{T}.$$

□

Remark 11.6. In general we have

$$\text{rank}(\mathcal{R}) - \text{rank}(\mathcal{T}) = N - \phi/2 - \lceil N/2 \rceil = \begin{cases} \frac{N-\phi}{2}, & N \text{ even} \\ \frac{N-\phi-1}{2}, & N \text{ odd} \end{cases}$$

This is thus the number of independent real polynomials we expect to find which do not correspond to pairing-off resonances.

In fact, for our purposes in this paper, we are interested in the additive group \mathcal{R}_0 of real polynomials with constant term zero. Theorem 11.4 can easily be amended to apply to this group:

Corollary 11.7. For any polynomial $\rho = \rho(x)$ we let $\tilde{\rho} := \rho(x) - \rho(0)$. Let B be any \mathbb{Z} -basis of \mathcal{K} . Let $\tilde{B} := \{\tilde{\rho} \mid \rho \in B\}$. Then

$$\tilde{B} \cup \{x - x^{N-1}, \dots, x^{\phi/2-1} - x^{N-\phi/2+1}\}$$

is a \mathbb{Z} -basis of the group \mathcal{R}_0 of real polynomials with zero constant term.

11.2.2. A basis for the kernel of the evaluation map

The results of this section have motivated the construction of 5-wave resonances via the cyclotomic method (Section 7). By Theorem 11.4, in order to find all real polynomials we must construct a basis of the kernel \mathcal{K} . Here is one such basis:

Proposition 11.8. Suppose that N is divisible by an odd prime. Then $\phi < N$. Let $F_i(x) := x^i \Phi_{2N}(x)$ for $i = 0, \dots, N - \phi - 1$. The set $\{F_i(x)\}_{i=0}^{N-\phi-1}$ is a basis for \mathcal{K} .

Proof. For each i , $F_i(\zeta) = \zeta^i \Phi_{2N}(\zeta) = 0$. Furthermore the degree of $F_i(x)$ is $\phi + i$. Thus the $F_i(x)$ belong to \mathcal{K} and they are linearly independent since their degrees all differ. Since there are $N - \phi$ of them and since they are monic polynomials, it follows that they form a basis of the kernel space. □

Remark 11.9. If, on the other hand, $N = 2^r$, then $N = \phi$ and $\mathcal{K} = 0$.

However, the polynomials $F_i(x)$ are generally complicated and long, since the general cyclotomic polynomial can be complicated and unpredictable. For this reason, we now construct a basis such that the polynomials belonging to it are particularly short and easy to write down explicitly. Each of them has length p_i for some prime p_i dividing N .

Before describing this basis of \mathcal{K} , we make some observations and introduce some notation:

Suppose that the prime p divides $2N$. Then $\zeta_p := \zeta^{2N/p}$ is a primitive p th root of unity and hence $0 = \Phi_p(\zeta_p) = \Phi_p(\zeta^{2N/p}) = 1 + \zeta^{2N/p} + \zeta^{2 \cdot (2N/p)} + \cdots + \zeta^{(p-1) \cdot (2N/p)}$. It follows that $\Phi_p(x^{2N/p}) \in \mathcal{K}$.

More generally, any polynomial in P of the form $x^s \Phi_p(x^{2N/p})$ will lie in \mathcal{K} . We will describe a basis for \mathcal{K} in terms of these polynomials. Note that if $\alpha = x^s \Phi_p(x^{2N/p})$ then $\deg(\alpha) = s + \frac{2N(p-1)}{p} \geq N$ so that $\alpha(x) \notin P$ since its degree is greater than $N - 1$. However, for convenience, we will interpret such a polynomial as an element of P in the following way: if the coefficient of x^k in α is nonzero, and if $k = tN + r$, with $r \leq N - 1$, then replace x^k with $(-1)^t x^r$. In fact, strictly speaking, we are working in the ring $\mathbb{Z}[x]/\langle x^N + 1 \rangle$ and choosing a lowest degree representative: see Appendix C.

Theorem 11.10. Suppose that $2N = 2^{b_0} p_1^{b_1} \cdots p_t^{b_t}$ where the p_i are distinct odd primes.

Let $x_i = x^{2N/p_i^{b_i}}$ for $i = 0, \dots, t$ (where $p_0 = 2$). Let $\Phi_i := \Phi_{p_i}(x^{2N/p_i})$.

Then $\mathcal{K} = \{0\}$ if N is a power of 2.

Otherwise, a \mathbb{Z} -basis of \mathcal{K} is $E = E_1 \cup \dots \cup E_t$ where

$$E_i := \{x_0^{m_0} x_1^{m_1} \dots x_t^{m_t} \Phi_i \mid 0 \leq m_j \leq N_{j,i}, \quad j = 0, \dots, t\}$$

where

$$N_{j,i} := \begin{cases} p_j^{b_j-1} - 1, & j = 0, i \\ p_j^{b_j} - 1, & 1 \leq j < i \\ p_j^{b_j} - p_j^{b_j-1} - 1, & i < j \leq t. \end{cases}$$

Proof. See [Appendix C](#) below.

As an immediate corollary to [Theorems 11.4](#) and [11.10](#) and [Corollary 11.7](#) we obtain: □

Corollary 11.11. Any real polynomial with zero constant term $\alpha(x) \in \mathcal{R}_0$ has a unique representation of the form

$$\alpha(x) = \sum_{p(x) \in E} \mu_p(p(x) - p(0)) + \sum_{k=1}^{\phi/2-1} \lambda_k(x^k - x^{N-k})$$

with $\mu_p, \lambda_k \in \mathbb{Z}$.

Remark 11.12. It is not hard to see that the polynomials Φ_i belong to the group \mathcal{T} of trivially real polynomials; i.e. their terms pair off.

However this is usually not the case for the polynomials $x^m \Phi_i$ for $m \geq 1$.

Example 11.13. We illustrate the theorem in the case $N = 15 = 3 \cdot 5 = p_1 \cdot p_2$.

Then $\Phi_1 = \Phi_3(x^{10}) = 1 + x^{10} + x^{20} = 1 - x^5 + x^{10}$. So

$$E_1 = \{\Phi_1, x_2 \Phi_1, x_2^2 \Phi_1, x_2^3 \Phi_1\}.$$

Here $x_2 = x^6$ and hence

$$\begin{aligned} x_2 \Phi_1 &= x^6 \Phi_1 = x^6 - x^{11} + x^{16} = x^6 - x^{11} - x \\ x_2^2 \Phi_1 &= x^{12} \Phi_1 = x^{12} - x^{17} + x^{22} = x^{12} + x^2 - x^7 \\ x_2^3 \Phi_1 &= x^{18} \Phi_1 = x^{18} - x^{23} + x^{28} = -x^3 + x^8 - x^{13} \end{aligned}$$

Thus

$$E_1 = \{1 - x^5 + x^{10}, x^6 - x^{11} - x, x^{12} + x^2 - x^7, -x^3 + x^8 - x^{13}\}.$$

Similarly, $\Phi_2 = \Phi_5(x^6) = 1 + x^6 + x^{12} + x^{18} + x^{24} = 1 + x^6 + x^{12} - x^3 - x^9$. We have

$$E_2 = \{\Phi_2, x_1 \Phi_2, x_1^2 \Phi_2\} = \{1 + x^6 + x^{12} - x^3 - x^9, x^{10} - x - x^7 - x^{13} + x^4, -x^5 - x^{11} + x^2 + x^8 + x^{14}\}.$$

Of course, our basis $E = E_1 \cup E_2$ consists of $N - \phi = 15 - 8 = 7$ polynomials.

Example 11.14. We will construct the simplest example of an irreducible resonance which is not a pairing-off resonance.

We take $N = 2 \cdot 3 = 6$ (so that $\text{rank}(\mathcal{R}) - \text{rank}(\mathcal{T}) = 1$). This is of course the smallest value of N which is neither prime nor a power of 2. Then $\phi = \phi(12) = 4$ and so \mathcal{K} has rank 2.

A basis for \mathcal{K} is $\{\Phi_1, x^3 \Phi_1\}$ where $\Phi_1 = 1 + x^4 + x^8 = 1 + x^4 - x^2$.

The first of these is trivially real. The second of these gives the real (indeed, vanishing) – but not trivially real – polynomial

$$\sum_{k=1}^5 \rho_k \zeta^k := \zeta - \zeta^3 + \zeta^5$$

which gives the resonance

$$(\zeta - \zeta^{-1}) + (\zeta^5 - \zeta^{-5}) = (\zeta^3 - \zeta^{-3})$$

with $M = 3$. However this does not satisfy the momentum conservation condition since $\sum_k k \rho_k = 3 \not\equiv 0 \pmod{6}$.

We multiply by 2 to obtain an irreducible resonance which is not a pairing-off resonance:

$$2(\zeta - \zeta^{-1}) + 2(\zeta^5 - \zeta^{-5}) = 2(\zeta^3 - \zeta^{-3}).$$

11.3. The case $N = 2p$

By [Corollary 11.5](#) above if N is either a power of 2 or a prime, all wave resonances are of pairing-off type and hence the total number of interacting waves M will always be even.

When $N = 2p$ where p is an odd prime, [Corollary 11.5](#) tells us that there are wave resonance solutions which are not of pairing-off type. Nevertheless, we can show that in this case also the total number of interacting waves M must be even.

Proposition 11.15. *Let $N = 2p$ where p is an odd prime number. Let $\rho(x) \in \mathcal{R}_0$ be a real polynomial with constant term 0. Suppose that $\rho'(1) \equiv 0 \pmod{N}$. Then the total number of interacting waves $M(\rho)$ is even.*

In other words, when $N = 2p$ then any wave resonance solution must have even total number of interacting waves M .

Proof. Let $\rho = \sum_{k=0}^d \rho_k \in \mathbb{Z}[x]$. For any integer m we have $m \equiv |m| \pmod{2}$. It follows that

$$M(\rho) = \sum_{k=1}^d |\rho_k| \equiv \sum_{k=1}^d \rho_k = \rho(1) \pmod{2}.$$

In particular, the map $\rho \mapsto M(\rho) \pmod{2}$ is linear.

Furthermore, note that if $k = N + s$ for some $s \geq 0$, replacing x^k by $-x^s$ does not affect $M(\rho) \equiv \rho(1) \pmod{2}$. Also, since N is even, $\rho'(1) \pmod{2}$ is unchanged.

Now suppose given a real polynomial $\rho = \sum_{k=1}^{N-1} \rho_k \in \mathcal{R}_0$. We *Claim* that

$$\rho(1) \equiv \rho'(1) \pmod{2}.$$

By linearity, it is enough to prove this claim on a basis of \mathcal{R}_0 . By [Corollary 11.7](#) and [Theorem 11.10](#) a basis is

$$\{x^i - x^{N-i} \mid i = 1, \dots, p-2\} \cup \{\Phi_1(x) - 1, x^p \Phi_1(x)\}$$

where $\Phi_1(x) = \Phi_p(x^4) = 1 + x^4 + \dots + x^{4(p-1)}$.

When $\rho = x^i - x^{N-i}$ then $\rho(1) = 0$ while $\rho'(1) = 2i - N \equiv 0 \pmod{2}$. When $\rho = \Phi_1(x) - 1$, $\rho(1) = p - 1 \equiv 0 \pmod{2}$ and $\rho'(1) = 4(\sum_{k=1}^{p-1} k) \equiv 0 \pmod{2}$. When $\rho = x^p \Phi_1(x)$, $\rho(1) = \Phi_1(1) = p \equiv 1 \pmod{2}$ and $\rho'(1) = \Phi_1'(1) + p \Phi_1(1) \equiv 0 + 1 \equiv 1 \pmod{2}$. This proves the *Claim*.

Thus if $\rho'(1) \equiv 0 \pmod{N}$ then $\rho'(1) \equiv 0 \pmod{2}$, since N is even. Thus $M(\rho) \equiv \rho(1) \equiv 0 \pmod{2}$ by the *Claim*. \square

Remark 11.16. If N is odd but not prime, there are always wave resonance solutions with odd wave number: There will be a basis vanishing polynomial of the form $\rho = x^i \Phi_p(x^k)$ with $i \geq 1$ and p a prime divisor of N . Then $M(\rho) \equiv \rho(1) \equiv p \equiv 1 \pmod{2}$ so that $M(\rho)$ is odd. Then $N\rho$ is a solution of the momentum conservation condition whose wave number $N \cdot M(\rho)$ is again odd.

Remark 11.17. The argument in the proof of [Proposition 11.15](#) does not extend to the case that $N = 2K$ where K is odd but not a prime. In this case $K = p_1^{b_1} \dots p_t^{b_t}$ where either $b_1 \geq 2$ or $t \geq 2$. Either way, by [Theorem 11.10](#) there is a basis element of \mathcal{K} of the form $\alpha(x) = x^i \Phi_p(x^k)$ where p is a prime dividing K and $i \geq 2$ is even. Then $\alpha'(1)$ is even but $\alpha(1) = p$ is odd. It follows that $\rho = K\alpha$ satisfies the momentum conservation condition $\rho'(1) \equiv 0 \pmod{N}$ and has odd wave number.

Remark 11.18. Likewise, the argument of [Proposition 11.15](#) does not extend to the case where $N = 4p$ with p an odd prime. In this case, by [Theorem 11.10](#) there is a basis polynomial of \mathcal{R}_0 of the form $\alpha(x) = x^{2p} \Phi_p(x^8)$. So $\alpha(1) = p$ but $\alpha'(1)$ is of the form $2p^2 + 8s \equiv 2 \pmod{4}$. If we let $\beta(x) = \alpha(x) + (x - x^{N-1})$ then $\beta(1)$ is odd, but $\beta'(1) = \alpha'(1) + (2 - N)$ is divisible by 4. Hence $\rho(x) = p\beta(x)$ satisfies the momentum conservation condition and has odd wave number $M(\rho)$.

We now find a simplified basis of \mathcal{R}_0 in the case $N = 2p$.

Lemma 11.19. *Let $m \geq 1$ be any odd number. Then*

$$(i) \ x^4 + x^8 + \dots + x^{4(m-1)} \equiv \sum_{i=1}^{\frac{m-1}{2}} (-1)^i (x^{2i} - x^{2(m-i)}) \pmod{x^{2m} + 1}.$$

$$(ii) \ x^{m+4} + x^{m+8} + \dots + x^{m+4(m-1)} \equiv (-1)^{\frac{m+1}{2}} \sum_{i=1}^{\frac{m-1}{2}} (-1)^i (x^{2i-1} + x^{2m-(2i-1)}) \pmod{x^{2m} + 1}.$$

Proof. Recall that for any $s \geq 1$, $x^s = (-1)^{\lfloor s/2m \rfloor} x^r \pmod{x^{2m} + 1}$ where $s = r \pmod{2m}$ and $0 \leq r < 2m$.

(i) If $t \leq \frac{m-1}{2}$ then $4t < 2m$ and $\lfloor 4t/2m \rfloor = 0$. In this case $x^{4t} = x^{2i}$ where $i = 2t$ is even.

If $\frac{m+1}{2} \leq t \leq m-1$ then $2m < 4t < 4m$ and $\lfloor 4t/2m \rfloor = 1$. In this case $x^{4t} = -x^{4t-2m} = -x^{2j} \pmod{x^{2m} + 1}$ where $j = 2t - m$ is odd. Thus $j = m - i$ where $i = 2(m - t)$ is even.

(ii) By (i),

$$x^{m+4} + x^{m+8} + \dots + x^{m+4(m-1)} = \sum_{i=1}^{\frac{m-1}{2}} (-1)^i x^m (x^{2i} - x^{2(m-i)}) = \sum_{i=1}^{\frac{m-1}{2}} (-1)^i (x^{m+2i} - x^{3m-2i}) \pmod{x^{2m} + 1}. \quad (56)$$

For $i \leq \frac{m-1}{2}$, $m + 2i < 2m$ and $x^{m+2i} = x^{2j-1}$ where $j = i + \frac{m+1}{2}$. It follows that $(-1)^i = (-1)^{\frac{m+1}{2}} (-1)^j$. Likewise, for $i \leq \frac{m-1}{2}$ we have $2m < 3m - 2i < 3m$ so that $x^{3m-2i} = -x^{m-2i} = -x^{2m-(2j-1)}$ where $j = i + \frac{m+1}{2}$. \square

Corollary 11.20. Let $N = 2p$ where p is an odd prime number. Then a basis for \mathcal{R}_0 is $\{x^i - x^{N-i}\}_{i=1}^{p-1} \cup \{\Psi(x)\}$ where

$$\Psi(x) = x^p + (-1)^{\frac{p-1}{2}} \sum_{i=1}^{\frac{p-1}{2}} (-1)^i 2x^{2i-1}. \quad (57)$$

Proof. By Corollary 11.11 a basis for \mathcal{R}_0 is $\{x^i - x^{2p-i}\}_{i=1}^{p-2} \cup \{\Phi - 1, x^p \Phi\}$ where $\Phi = \Phi_p(x^4) = 1 + x^4 + \dots + x^{4(p-1)}$. By Lemma 11.19 (i),

$$\Phi - 1 = \sum_{i=1}^{\frac{p-1}{2}} (-1)^i (x^{2i} - x^{2p-2i}). \quad (58)$$

We can replace $\Phi - 1$ with $(\Phi - 1) - \sum_{i=1}^{\frac{p-3}{2}} (-1)^i (x^{2i} - x^{2p-2i}) = (-1)^{\frac{p-1}{2}} (x^{p-1} - x^{p+1})$.

By Lemma 11.19 (ii), we have

$$x^p \Phi = x^p + (-1)^{\frac{p+1}{2}} \sum_{i=1}^{\frac{p-1}{2}} (-1)^i (x^{2i-1} + x^{2p-(2i-1)}). \quad (59)$$

We can thus replace $x^p \Phi$ with $x^p \Phi + (-1)^{\frac{p+1}{2}} \sum_{i=1}^{\frac{p-1}{2}} (-1)^i (x^{2i-1} - x^{2p-(2i-1)}) = \Psi$. \square

Proposition 11.21. Let $N = 2p$ where p is an odd prime number. Let $\rho(x) \in \mathcal{R}_0$ satisfy $\rho'(1) \equiv 0 \pmod{N}$. If ρ is not trivially real then $M(\rho) \geq N$.

In other words, when $N = 2p$ any wave resonance solution which is not of pairing-off type has a total of at least N interacting waves.

Proof. If $\rho \in \mathcal{R}_0$ then Corollary 11.20 tells us that ρ is uniquely expressible in the form $\rho = \sum_{i=1}^{p-1} \lambda_i (x^i - x^{N-i}) + \mu \Psi(x)$. Thus

$$\begin{aligned} M(\rho) &= |\mu| + \sum_{i=1}^{\frac{p-1}{2}} (|\lambda_{2i-1} \pm 2\mu| + |\lambda_{2i-1}|) + \sum_{i=1}^{\frac{p-1}{2}} 2|\lambda_{2i}| \\ &\geq |\mu| + \sum_{i=1}^{\frac{p-1}{2}} (|\lambda_{2i-1} \pm 2\mu| + |\lambda_{2i-1}|) \\ &\geq |\mu| + \frac{p-1}{2} |2\mu| = p|\mu|. \end{aligned}$$

If $\rho'(1) \equiv 0 \pmod{N}$ then Proposition 11.15 tells us that $M(\rho)$ is even. But $M(\rho) \equiv \rho(1) \equiv \mu \pmod{2}$. Thus μ is even.

Now ρ is trivially real if and only if $\mu = 0$. Therefore, if ρ is not trivially real we have $|\mu| \geq 2$ and therefore $M(\rho) \geq p|\mu| \geq 2p = N$. \square

Remark 11.22. On the other hand, whenever $N = 2K$ there are many pairing-off 4-wave resonances, such as $\rho = (x - x^{N-1}) + (x^{K-1} - x^{K+1})$ (see Section 4).

12. Conclusions

The numerical results observed by Fermi, Pasta, Ulam and Tsingou have triggered the discovery of new scientific paths, such as soliton dynamics and integrability in partial differential equations. However, the explanation of the numerical observations has remained elusive, and still is in some aspects, after over 50 years. In this paper we have concentrated on the role played by exact resonant interactions on the dynamics that leads to thermalisation; at the moment such a role is only a conjecture, but it finds a solid foundation in the limit of weak nonlinearity via the Wave Turbulence Theory which is formally developed in the thermodynamic limit. In discrete low-dimensional chains the role played by exact discrete resonances seems to be very important (see [8,13,14]); the results contained in these papers have stimulated us to investigate the possible discrete resonances allowed by the linear dispersion relation of the α - and β -FPUT systems. In this regard we show that N , the number of particles, plays a crucial role via the set of its divisors. Up to date this result has remained unnoticed, possibly because, for computational reasons, most of the simulations in the literature have been performed with N equal to a power of 2. For extensive numerical/analytical studies over many values of N (all of them powers of two) see, amongst others, [10,24,25] and references therein.

The results and the methodology used in the present paper are very articulated: Table 2 shows an overview of the most important results obtained. The 6-wave resonant interactions play an important role for many values of N . However, it turns out that, for some specific values of N (namely when $N > 6$ and N is divisible by 3), 5-wave resonances do exist and they may in principle lead to fast equipartition due to common-mode connections amongst 5-wave resonances, forming a web of

interconnected modes, which we term super-cluster ([Definition 9.1](#)). A quantitative assessment of this proposed equipartition using numerical simulations of the FPUT system is still unexplored. It would be interesting to observe the fast 5-wave dominated dynamics as opposed to the slow 6-wave dynamics, in order to gain insights on the route to thermalisation. Also, it would be interesting to study whether the multi-component nature of the super-cluster is evident in numerical simulations. Theoretically, one important study concerns the calculation of the maximal number of conservation laws (apart from the energy) for a given component of the super-cluster. Such extra conservation laws are known to arise, at any order, in a variety of resonant systems such as the Charney-Hasegawa-Mima equation, see [\[26\]](#).

Finally, from a number-theoretical point of view there is still an open question: Does the cyclotomic method described in [Section 7](#), to construct 5-wave resonances out of real FPUT polynomials of length 3, fulfil all possible 5-wave resonances? We have checked by brute-force search that the answer is positive for $N \leq 1155 (= 3 \cdot 5 \cdot 7 \cdot 11)$, but for larger N the wild nature of cyclotomic polynomials as the number of divisors of N grows does not rule out the possibility that real FPUT polynomials of various lengths may combine to produce new resonant FPUT polynomials of length 5.

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Appendix A. Interaction matrix

After removing non-resonant three wave interactions, the matrix in the Zakharov equation is given by (see [\[15\]](#)):

$$\begin{aligned} \bar{T}_{1,2,3,4} = & T_{1,2,3,4} - V_{1,3,1-3}V_{4,2,4-2} \left[\frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4} \right] \\ & - V_{2,3,2-3}V_{4,1,4-1} \left[\frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4} \right] \\ & - V_{1,4,1-4}V_{3,2,3-2} \left[\frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3} \right] \\ & - V_{2,4,2-4}V_{3,1,3-1} \left[\frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3} \right] \\ & - V_{1+2,1,2}V_{3+4,3,4} \left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4} \right] \\ & - V_{-1-2,1,2}V_{-3-4,3,4} \left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4} \right]. \end{aligned} \quad (\text{A.1})$$

Appendix B. Constructing 6-wave resonances for arbitrary $N (\geq 6)$, which are not of pairing-off form

We provide results of combining sums or subtractions of two real FPUT polynomials of length 3, to obtain FPUT resonant polynomials of length six, i.e. 6-wave resonances or, in short, sextuplets. These are not of pairing-off form so they correspond to truly different resonances with respect to the ones discussed in [Section 4.3](#). As in the case of 5-wave resonances, there will be a number of relevant cases.

B1. Case 0: $3 \nmid N$.

In this case there are no 6-wave resonances (which are not of pairing-off form) from our method.

B2. Case 1: $3|N$ and $N \geq 6$.

In this case there are 5 one-parameter families of sextuplets. These are defined in [Table B.3](#) below.

Regarding ‘complex conjugation’ operation, i.e. $k_j \rightarrow N - k_j$, $j = 1, \dots, 6$ (related to pairing-off terms), notice that the sextuplets in Family 1 are invariant under this operation. The other families’ sextuplets come in conjugate pairs.

Regarding the clustering of the sextuplets in these families, we do not provide a thorough discussion but just mention the following: (i) By simple wavenumber counting, within a given family we expect very few connections in families 1, 4 and 5. (ii) By looking at their explicit parameterisations, it is evident that families 1, 2 and 3 can be grouped via 3-common-mode connections, and families 4 and 5 can be grouped via 3-common-mode connections as well. Connections between these groups of families 1-2-3 and families 4-5 are not evident from the parameterisation but they occur too, and they range from 2- to 4-common-mode connections.

We consider some examples:

Table B1

Summary of 6-wave resonances obtained via linear combinations of two real FPUT polynomials of length 3.

	Form of Sextuplets: $\{k_1, \dots, k_S; k_{S+1}, \dots, k_{S+T}\}$	Comments
Family 1	$\{n, \frac{N}{3} - n, n + \frac{2N}{3}, N - n; n + \frac{N}{3}, \frac{2N}{3} - n\},$ $n = 1, \dots, \lfloor N/6 \rfloor$	$S = 4, T = 2$ $3 N \wedge N \geq 6$ Total Sextuplets: $\lfloor N/6 \rfloor$.
Family 2	$\{n, \frac{N}{3} - 3n, 3n, n + \frac{2N}{3}; 3n + \frac{N}{3}, \frac{2N}{3} - n\},$ $\{N - n, 3n + \frac{2N}{3}, N - 3n, \frac{N}{3} - n; \frac{2N}{3} - 3n, n + \frac{N}{3}\},$ $n = 1, \dots, \lfloor N/9 \rfloor - 1$	$S = 4, T = 2$ $3 N \wedge N \geq 12$ Complex-conjugate pairs: $k_j \rightarrow N - k_j, j = 1, \dots, 6.$ Total Sextuplets: $2\lfloor N/9 \rfloor - 2.$
Family 3	$\{n, \frac{2N}{3} - 3n, n + \frac{2N}{3}; \frac{2N}{3} - n, N - 3n, 3n + \frac{2N}{3}\},$ $\{N - n, 3n + \frac{N}{3}, \frac{N}{3} - n; n + \frac{N}{3}, 3n, \frac{N}{3} - 3n\},$ $n = 1, \dots, \lfloor N/9 \rfloor - 1, \quad n \neq N/12$	$S = 3, T = 3$ $3 N \wedge N \geq 15$ Complex-conjugate pairs: $k_j \rightarrow N - k_j, j = 1, \dots, 6.$ Total Sextuplets: $2\lfloor N/9 \rfloor - 2.$ (2 Sextuplets less if $12 N$)
Family 4	$\{3n - \frac{N}{3}, \frac{N}{3} - n, 3n + \frac{N}{3}, N - n; 3n, n + \frac{N}{3}\},$ $\{\frac{4N}{3} - 3n, \frac{2N}{3} + n, \frac{2N}{3} - 3n, n; N - 3n, \frac{2N}{3} - n\},$ $n = \lfloor N/9 \rfloor + 1, \dots, \lfloor N/6 \rfloor - 1$	$S = 4, T = 2$ $3 N \wedge N \geq 15 \wedge N \neq 18$ Complex-conjugate pairs: $k_j \rightarrow N - k_j, j = 1, \dots, 6.$ Total Sextuplets: $2\lfloor N/6 \rfloor - 2\lfloor N/9 \rfloor - 2.$
Family 5	$\{3n - \frac{N}{3}, \frac{2N}{3} - n, 3n + \frac{N}{3}; n, 3n, n + \frac{2N}{3}\},$ $\{\frac{4N}{3} - 3n, \frac{N}{3} + n, \frac{2N}{3} - 3n; N - n, N - 3n, \frac{N}{3} - n\},$ $n = \lfloor N/9 \rfloor + 1, \dots, \lfloor N/6 \rfloor - 1$	$S = 3, T = 3$ $3 N \wedge N \geq 15 \wedge N \neq 18$ Complex-conjugate pairs: $k_j \rightarrow N - k_j, j = 1, \dots, 6.$ Total Sextuplets: $2\lfloor N/6 \rfloor - 2\lfloor N/9 \rfloor - 2.$

Example B.1. $N = 6$ or $N = 9$. In this case we have only one sextuplet, coming from Family 1. Explicitly, when $N = 6$ we get the sextuplet $\{1, 1, 5, 5; 3, 3\}$, already discussed in an earlier Section. The new case is $N = 9$, which gives the sextuplet $\{1, 2, 7, 8; 4, 5\}$, corresponding to the identity $\sin(\pi/9) + \sin(2\pi/9) = \sin(4\pi/9)$. It is remarkable that an ‘incomplete resonant triad’ (i.e., a triad where frequency resonance is satisfied but momentum condition is not) is always involved in these resonances.

Example B.2. $N = 12$. In this case Family 1 provides the sextuplets

$$n = 1 : \quad \{1, 3, 9, 11; 5, 7\},$$

$$n = 2 : \quad \{2, 2, 10, 10; 6, 6\}.$$

Family 2 provides sextuplets in conjugate pairs:

$$\{1, 1, 3, 9; 7, 7\} \quad \text{and} \quad \{11, 11, 9, 3; 5, 5\}.$$

We now present and discuss the construction of exceptional sextuplets, made by combining cyclotomic polynomials of length 3 and length 5, in such a way that one monomial from each polynomial cancel out, resulting in a FPUT resonant polynomial of length 6.

Unlike the sextuplets obtained above, these exceptional sextuplets do not come in one-parameter families; rather, there is a fixed number of them depending on the divisibility of the number of particles N .

B.3. Case 2: $15|N$ and $2 \nmid N$.

In this case, in addition to the five families of sextuplets summarised in Table B.3, there are four conjugate pairs of resonant sextuplets. The first two pairs convert 4 to 2 waves and the last two pairs convert 3 to 3 waves:

$$\begin{aligned} &\left\{ \frac{N}{15}, \frac{N}{15}, \frac{2N}{15}, \frac{7N}{15}, \frac{N}{3}, \frac{2N}{5} \right\} && \left\{ \frac{14N}{15}, \frac{14N}{15}, \frac{13N}{15}, \frac{8N}{15}, \frac{2N}{3}, \frac{3N}{5} \right\} \\ &\left\{ \frac{N}{15}, \frac{4N}{5}, \frac{13N}{15}, \frac{13N}{15}, \frac{N}{3}, \frac{4N}{15} \right\} && \left\{ \frac{14N}{15}, \frac{N}{5}, \frac{2N}{15}, \frac{2N}{15}, \frac{2N}{3}, \frac{11N}{15} \right\} \\ &\left\{ \frac{N}{15}, \frac{8N}{15}, \frac{8N}{15}, \frac{N}{5}, \frac{2N}{3}, \frac{4N}{15} \right\} && \left\{ \frac{14N}{15}, \frac{7N}{15}, \frac{7N}{15}, \frac{N}{3}, \frac{4N}{5}, \frac{11N}{15} \right\} \end{aligned}$$

$$\left\{ \frac{3N}{5}, \frac{2N}{15}, \frac{7N}{15}, \frac{2N}{3}, \frac{4N}{15}, \frac{4N}{15} \right\} \quad \left\{ \frac{2N}{5}, \frac{13N}{15}, \frac{8N}{15}, \frac{N}{3}, \frac{11N}{15}, \frac{11N}{15} \right\}. \quad (\text{B.1})$$

B.4. Case 3: $30|N$ and $4 \nmid N$.

In this case the four conjugate pairs of sextuplets from Eq. (B.1) still apply, but there are additional sextuplets: one sextuplet that is invariant under complex conjugation and converts 4 to 2 waves:

$$\left\{ \frac{N}{10}, \frac{N}{6}, \frac{5N}{6}, \frac{9N}{10}, \frac{3N}{10}, \frac{7N}{10} \right\}, \quad (\text{B.2})$$

and seven conjugate pairs of sextuplets, of which three convert 4 to 2 waves and four convert 3 to 3 waves:

$$\begin{aligned} & \left\{ \frac{N}{30}, \frac{13N}{30}, \frac{5N}{6}, \frac{9N}{10}, \frac{17N}{30}, \frac{19N}{30} \right\} \quad \left\{ \frac{29N}{30}, \frac{17N}{30}, \frac{N}{6}, \frac{N}{10}, \frac{13N}{30}, \frac{11N}{30} \right\} \\ & \left\{ \frac{N}{30}, \frac{N}{6}, \frac{23N}{30}, \frac{9N}{10}, \frac{7N}{30}, \frac{19N}{30} \right\} \quad \left\{ \frac{29N}{30}, \frac{5N}{6}, \frac{7N}{30}, \frac{N}{10}, \frac{23N}{30}, \frac{11N}{30} \right\} \\ & \left\{ \frac{N}{30}, \frac{N}{2}, \frac{9N}{10}, \frac{9N}{10}, \frac{19N}{30}, \frac{7N}{10} \right\} \quad \left\{ \frac{29N}{30}, \frac{N}{2}, \frac{N}{10}, \frac{N}{10}, \frac{11N}{30}, \frac{3N}{10} \right\} \\ & \left\{ \frac{N}{30}, \frac{13N}{30}, \frac{N}{2}, \frac{11N}{30}, \frac{23N}{30}, \frac{5N}{6} \right\} \quad \left\{ \frac{29N}{30}, \frac{17N}{30}, \frac{N}{2}, \frac{19N}{30}, \frac{7N}{30}, \frac{N}{6} \right\} \\ & \left\{ \frac{N}{30}, \frac{7N}{30}, \frac{3N}{10}, \frac{N}{6}, \frac{13N}{30}, \frac{29N}{30} \right\} \quad \left\{ \frac{29N}{30}, \frac{23N}{30}, \frac{7N}{10}, \frac{5N}{6}, \frac{17N}{30}, \frac{N}{30} \right\} \\ & \left\{ \frac{N}{10}, \frac{N}{2}, \frac{17N}{30}, \frac{7N}{10}, \frac{7N}{10}, \frac{23N}{30} \right\} \quad \left\{ \frac{9N}{10}, \frac{N}{2}, \frac{13N}{30}, \frac{3N}{10}, \frac{3N}{10}, \frac{7N}{30} \right\} \\ & \left\{ \frac{N}{6}, \frac{11N}{30}, \frac{17N}{30}, \frac{19N}{30}, \frac{7N}{10}, \frac{23N}{30} \right\} \quad \left\{ \frac{5N}{6}, \frac{19N}{30}, \frac{13N}{30}, \frac{11N}{30}, \frac{3N}{10}, \frac{7N}{30} \right\}. \end{aligned} \quad (\text{B.3})$$

B.5. Case 4: $60|N$.

In this case all sextuplets from Eqs. (B.1)–(B.3) still apply, but there are four additional conjugate pairs of sextuplets, all converting 3 to 3 waves:

$$\begin{aligned} & \left\{ \frac{N}{60}, \frac{41N}{60}, \frac{3N}{4}, \frac{N}{20}, \frac{11N}{20}, \frac{17N}{20} \right\} \quad \left\{ \frac{59N}{60}, \frac{19N}{60}, \frac{N}{4}, \frac{19N}{20}, \frac{9N}{20}, \frac{3N}{20} \right\} \\ & \left\{ \frac{N}{20}, \frac{29N}{60}, \frac{11N}{20}, \frac{11N}{60}, \frac{N}{4}, \frac{13N}{20} \right\} \quad \left\{ \frac{19N}{20}, \frac{31N}{60}, \frac{9N}{20}, \frac{49N}{60}, \frac{3N}{4}, \frac{7N}{20} \right\} \\ & \left\{ \frac{7N}{60}, \frac{31N}{60}, \frac{7N}{12}, \frac{N}{4}, \frac{17N}{60}, \frac{41N}{60} \right\} \quad \left\{ \frac{53N}{60}, \frac{29N}{60}, \frac{5N}{12}, \frac{3N}{4}, \frac{43N}{60}, \frac{19N}{60} \right\} \\ & \left\{ \frac{3N}{20}, \frac{11N}{20}, \frac{37N}{60}, \frac{N}{4}, \frac{7N}{20}, \frac{43N}{60} \right\} \quad \left\{ \frac{17N}{20}, \frac{9N}{20}, \frac{23N}{60}, \frac{3N}{4}, \frac{13N}{20}, \frac{17N}{60} \right\}. \end{aligned} \quad (\text{B.4})$$

Appendix C. Basis for vanishing sums: proof of Theorem 11.10

The approach here is motivated by the arguments of Lam and Leung [27]. An additive group A is *free* with basis $E = \{e_1, \dots, e_r\} \subset A$ if every element in A is uniquely expressible as a sum $\sum_{i=1}^r m_i e_i$ with $m_i \in \mathbb{Z}$. Here r is the *rank* of A . If E decomposes as a disjoint union $E_1 \cup E_2$ then $A = A_1 \oplus A_2$ where each A_i is free with basis E_i .

If $f(x) \in \mathbb{Z}[x]$ is a polynomial with integer coefficients then we can form the ring $\mathbb{Z}[x]/\langle f(x) \rangle$ whose elements are polynomials with integer coefficients and we identify the polynomials $h(x)$ and $g(x)$ whenever $h(x) = g(x) + f(x)m(x)$ for some $m(x) \in \mathbb{Z}[x]$.

If $f(x)$ is *monic* of degree d , then $\mathbb{Z}[x]/\langle f(x) \rangle$ is easily seen to be a free additive group with basis $\{1, x, \dots, x^{d-1}\}$.

Now for any $n \geq 1$ we let C_n denote the subring of \mathbb{C} consisting of all polynomials in $\zeta_n := e^{2\pi i/n}$. Then the natural map $\mathbb{Z}[x]/\langle \Phi_n(x) \rangle \rightarrow C_n$ sending $p(x)$ to $p(\zeta_n)$ is an isomorphism of rings: it sends the basis $\{1, x, \dots, x^{\phi(n)-1}\}$ to the basis $\{1, \zeta_n, \dots, \zeta_n^{\phi(n)-1}\}$. (See Theorem 11.2 (i).)

For any $n \geq 1$ we let P_n denote the ring $\mathbb{Z}[x]/\langle x^n - 1 \rangle$ and we let $D_n = \mathbb{Z}[x]/\langle x^n + 1 \rangle$. Note that $\{1, x, \dots, x^{n-1}\}$ is a basis in each case and that $D_n \cong P_n$ (via $x \mapsto -x$) when n is odd. Furthermore, $D_{2^n} = C_{2^{n+1}}$ since $\Phi_{2^{n+1}}(x) = x^{2^n} + 1$.

A subgroup B of a free group A is always free, and $\text{rank}(B) \leq \text{rank}(A)$, but – unlike the situation for vector spaces – B is not usually a direct summand of A . For example, if A is free with basis $\{e_1, \dots, e_r\}$ then $B := 2A = \{2a \mid a \in A\}$ is a free subgroup with basis $\{2e_1, \dots, 2e_r\}$, but is not a direct summand.

However, we do have the following:

Lemma C.1. Let $\phi: A \rightarrow B$ be a surjective homomorphism of free additive groups. Then there exists a submodule Q such that $A = \ker(\phi) \oplus Q$.

Proof. Let $\{e_1, \dots, e_s\}$ be a basis of $\ker(\phi)$. Let $\{f_1, \dots, f_t\}$ be a basis of B . Let e_{s+1}, \dots, e_{s+t} be any elements of A satisfying $\phi(e_{s+j}) = f_j$ for $1 \leq j \leq t$.

Then $E = \{e_1, \dots, e_{s+t}\}$ is a basis of A : Let $a \in A$. Then $\phi(a) = \sum_{j=1}^t m_j f_j$ for some (unique) $m_j \in \mathbb{Z}$. So $\phi(a) = \phi(\sum_{j=1}^t m_j e_{s+j})$ and thus $a - \sum_{j=1}^t m_j e_{s+j} \in \ker(\phi)$. Hence $a - \sum_{j=1}^t m_j e_{s+j} = \sum_{i=1}^s n_i e_i$ for some (unique) $n_i \in \mathbb{Z}$ and so $a = \sum_{i=1}^s n_i e_i + \sum_{j=1}^t m_j e_{s+j}$.

It follows that $A = \ker(\phi) \oplus Q$ where Q is free with basis $\{e_{s+1}, \dots, e_{s+t}\}$. \square

Lemma C.2. Let π_n denote the surjective homomorphism $P_n \rightarrow C_n$ sending $p(x)$ to $p(\zeta_n)$. Then

(i) $\ker(\pi_n)$ has basis $\{x^i \Phi_n(x)\}_{i=0}^{n-\phi(n)-1}$.

(ii) $P_n = \ker(\pi_n) \oplus Q_n$ where Q_n has basis $\{1, x, \dots, x^{\phi(n)-1}\}$.

Proof. (i) Let $p(x) \in \ker(\pi_n)$. We can assume that $p(x)$ has degree at most $n-1$. Then $p(\zeta_n) = 0$ and hence $p(x) = \Phi_n(x)r(x)$ for some $r(x) \in \mathbb{Z}[x]$. Clearly $r(x)$ has degree at most $n - \phi(n) - 1$. So $r(x) = \sum_{i=1}^{n-\phi(n)-1} m_i x^i$ and hence $p(x) = \sum_{i=1}^{n-\phi(n)-1} m_i x^i \Phi_n(x)$. This expression is unique since the degrees of the $x^i \Phi_n(x) \in P_n$ are all distinct.

(ii) Since $\pi_n(x^i) = \zeta_n^i$ for $i \leq \phi(n) - 1$, the proof of Lemma Appendix C.1 implies the result. \square

For any additive groups A and B , the tensor product $A \otimes B$ is defined. If A and B are rings, so is $A \otimes B$.

If A and B are free additive groups with bases $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_t\}$ respectively, then $A \otimes B$ is the free additive group with basis $\{e_i \otimes f_j\}$. If $a = \sum_i m_i e_i \in A$ and $b = \sum_j n_j f_j \in B$ then $a \otimes b$ denotes the element $\sum_{i,j} m_i n_j (e_i \otimes f_j) \in A \otimes B$. In particular, $A \otimes B$ is free of rank $\text{rank}(A) \cdot \text{rank}(B)$.

If $\phi: A \rightarrow A'$ and $\psi: B \rightarrow B'$ are homomorphisms (of groups or rings), there is a well-defined homomorphism $\phi \otimes \psi: A \otimes B \rightarrow A' \otimes B'$ sending $a \otimes b$ to $\phi(a) \otimes \psi(b)$.

Proposition C.3. Let $n, m \geq 1$ be relatively prime integers. Then the following are isomorphisms of rings:

(i) $P_n \otimes P_m \rightarrow P_{nm}$, $p(x) \otimes q(x) \rightarrow p(x^m)q(x^n)$

(ii) $D_n \otimes D_m \rightarrow D_{nm}$, $p(x) \otimes q(x) \rightarrow p(x^m)q(x^n)$

(iii) $C_n \otimes C_m \rightarrow C_{nm}$, $p(\zeta_n) \otimes q(\zeta_m) \rightarrow p(\zeta_{nm}^m)q(\zeta_{nm}^n)$.

Proof. One easily verifies that the maps described (when extended linearly to any element of the domain) are ring homomorphisms.

(i) This map sends the basis elements $x^i \otimes x^j$ of $P_n \otimes P_m$ to the elements x^{mi+nj} of P_{nm} . In P_{nm} , $x^{mi+nj} = x^r$ where $r < nm - 1$ and r is the remainder of $mi + nj$ on division by nm .

Suppose that $mi + nj = mi' + nj'$ for some $0 \leq i, i' \leq n-1$, $0 \leq j, j' \leq m-1$. Then $m(i-i') = n(j'-j)$. Thus n divides $m(i-i')$ and since $\gcd(n, m) = 1$ it follows that n divides $i-i'$. Since $0 \leq i, i' \leq n-1$ this forces $i = i'$. Similarly $j = j'$.

So the induced map from the basis $\{x^i \otimes x^j\}$ of $P_n \otimes P_m$ to the basis $\{x^r \mid 0 \leq r \leq nm - 1\}$ is injective and hence bijective. It follows that our ring homomorphism is an isomorphism.

(ii) An almost identical argument applies in this case: the only difference is that $x^{mi+nj} = \pm x^r$ in D_{nm} (the sign depending on the parity of $\lfloor \frac{mi+nj}{nm} \rfloor$).

(iii) This is a less obvious fact, but is well known from the theory of cyclotomic integers in number theory. For a proof, see for example [CITE Marcus, Number Fields, p36]. \square

Combining this result with an induction argument we deduce:

Corollary C.4. Let $n = p_1^{a_1} \dots p_t^{a_t}$ where $p_1 < \dots < p_t$ are primes. Let $n_i = n/p_i^{a_i}$ for $1 \leq i \leq t$. Then there are ring isomorphisms

$$1. P_{p_1^{a_1}} \otimes \dots \otimes P_{p_t^{a_t}} \cong P_n, \quad p_1(x) \otimes \dots \otimes p_t(x) \mapsto p_1(x^{n_1}) \dots p_t(x^{n_t}).$$

$$2. D_{p_1^{a_1}} \otimes \dots \otimes D_{p_t^{a_t}} \cong D_n, \quad p_1(x) \otimes \dots \otimes p_t(x) \mapsto p_1(x^{n_1}) \dots p_t(x^{n_t}).$$

$$3. C_{p_1^{a_1}} \otimes \dots \otimes C_{p_t^{a_t}} \cong C_n, \quad p_1(\zeta_{p_1^{a_1}}) \otimes \dots \otimes p_t(\zeta_{p_t^{a_t}}) \mapsto p_1(\zeta_{n_1}^{n_1}) \dots p_t(\zeta_{n_t}^{n_t}).$$

Now let $n = 2N = 2^{b_0} p_1^{b_1} \dots p_t^{b_t}$ where $2 < p_1 < \dots < p_t$ are primes (and $b_0 \geq 1$, of course). Thus $D_N = \mathbb{Z}[x]/(x^N + 1)$ is the ring of polynomials modulo the relation $x^k = (-1)^t x^r$ where $k = tN + r$.

Our goal is to calculate a basis for the kernel of the homomorphism $\rho_n: D_N \rightarrow C_n$ sending $p(x)$ to $p(\zeta)$ where $\zeta = \zeta_n = e^{\pi i/N}$.

By Corollary Appendix C.4 we have tensor decompositions

$$D_N \cong D_{2^{b_0-1}} \otimes P_{p_1^{b_1}} \otimes \dots \otimes P_{p_t^{b_t}}$$

and

$$C_n \cong C_{2^{b_0}} \otimes C_{p_1^{b_1}} \otimes \dots \otimes C_{p_t^{b_t}}.$$

With these identifications the map $\rho_n: D_N \rightarrow C_n$ becomes the map

$$D_{2^{b_0-1}} \otimes P_{p_1^{b_1}} \otimes \dots \otimes P_{p_t^{b_t}} \rightarrow C_{2^{b_0}} \otimes C_{p_1^{b_1}} \otimes \dots \otimes C_{p_t^{b_t}}$$

given by the formula

$$p_0(x) \otimes p_1(x) \otimes \cdots \otimes p_t(x) \mapsto p_0(\zeta_{2^{b_0}}) \otimes p_1(-\zeta_{p_1^{b_1}}) \otimes \cdots \otimes p_t(-\zeta_{p_t^{b_t}}).$$

Up to a sign, which does not affect the kernel, this is just the map $\rho_{2^{b_0}} \otimes \pi_{p_1^{b_1}} \otimes \cdots \otimes \pi_{p_t^{b_t}}$.

Lemma C.5. Let $\phi_1: A_1 \rightarrow B_1$ and $\phi_2: A_2 \rightarrow B_2$ be surjective homomorphisms of free additive groups. Suppose that $A_i = \ker(\phi_i) \oplus Q_i$ for $i = 1, 2$. Then

$$\ker(\phi_1 \otimes \phi_2) = (\ker(\phi_1) \otimes Q_2) \oplus (A_1 \otimes \ker(\phi_2))$$

and

$$A_1 \otimes A_2 = \ker(\phi_1 \otimes \phi_2) \oplus (Q_1 \otimes Q_2).$$

Proof. Let $r_i = \text{rank}(A_i)$, $s_i = \text{rank}(B_i)$ and $t_i = r_i - s_i = \text{rank} \ker(\phi_i)$ for $i = 1, 2$. Let $\{e_1, \dots, e_{t_1}\}$ and $\{f_1, \dots, f_{t_2}\}$ be bases of $\ker(\phi_1)$ and $\ker(\phi_2)$ respectively. Let $\{e_{t_1+1}, \dots, e_{r_1}\}$ and $\{f_{t_2+1}, \dots, f_{r_2}\}$ be bases of Q_1 and Q_2 respectively.

Let

$$x = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} m_{ij}(e_i \otimes f_j) \in \ker(\phi_1 \otimes \phi_2).$$

Then

$$y := \sum_{i=1}^{r_1} \sum_{j=1}^{t_2} m_{ij}(e_i \otimes f_j) \in \ker(\phi_1 \otimes \phi_2)$$

since $f_j \in \ker(\phi_2)$ for $j \leq t_2$. Note that $y \in A_1 \otimes \ker(\phi_2)$.

Thus

$$z = x - y = \sum_{i=1}^{r_1} \sum_{j=t_2+1}^{r_2} m_{ij}(e_i \otimes f_j)$$

lies in $\ker(\phi_1 \otimes \phi_2)$. We rewrite this as

$$z = \sum_{j=t_2+1}^{r_2} w_j \otimes f_j$$

where $w_j = \sum_{i=1}^{r_1} m_{ij}e_i \in A_1$ for $j \geq t_2 + 1$. Thus

$$0 = (\phi_1 \otimes \phi_2)(z) = \sum_{j=t_2+1}^{r_2} \phi_1(w_j) \otimes \phi_2(f_j).$$

Since $\{\phi_2(f_j) \mid j = t_2 + 1, \dots, r_2\}$ is a basis of B_2 , it follows that $\phi(w_j) = 0$ for all $j \geq t_2 + 1$. Thus $w_j \in \ker(\phi_1)$ for $j \geq t_2 + 1$. It follows that $w_j = \sum_{i=1}^{t_1} m_{ij}e_i \otimes f_j$ and that $z \in \ker(\phi_1) \otimes Q_2$. This proves the first statement.

The second statement follows immediately from the proof of Lemma [Appendix C.1](#) since $\{(\phi_1 \otimes \phi_2)(e_i \otimes f_j)\}_{i > t_1, j > t_2} = \{\phi_1(e_i) \otimes \phi_2(f_j)\}_{i > t_1, j > t_2}$ is a basis of $B_1 \otimes B_2$. \square

Combining this lemma with a straightforward induction argument we deduce:

Corollary C.6. Let $\phi_i: A_i \rightarrow B_i$ be surjective homomorphisms of free additive groups, for $i = 1, \dots, t$. Suppose that $A_i = \ker(\phi_i) \oplus Q_i$ for each i .

Then $\ker(\phi_1 \otimes \cdots \otimes \phi_t) = \oplus_{i=1}^t K_i$ where

$$K_i = A_1 \otimes \cdots \otimes A_{i-1} \otimes \ker(\phi_i) \otimes Q_{i+1} \otimes \cdots \otimes Q_t.$$

Thus let F_i be a basis of A_i for each i and suppose that $F_i = K_i \cup G_i$ where K_i is a basis of $\ker(\phi_i)$ and G_i is a basis of Q_i . Then $E = E_1 \cup \cdots \cup E_t$ is a basis of $\ker(\phi_1 \otimes \cdots \otimes \phi_t)$ where

$$E_i = \left\{ x_1 \otimes \cdots \otimes x_t \mid x_j \in \begin{cases} E_j, & j < i \\ K_i, & j = i \\ G_j, & i < j \leq t \end{cases} \right\}.$$

Combining all of this we get: *Proof of Theorem 11.10.* Under the isomorphism

$$D_{2^{b_0-1}} \otimes P_{p_1^{b_1}} \otimes \cdots \otimes P_{p_t^{b_t}} \cong D_N$$

the kernel of $\rho_{2N}: D_N \rightarrow C_{2N}$ corresponds to $\ker(\rho_{2^{b_0}} \otimes \pi_{p_1^{b_1}} \otimes \cdots \otimes \pi_{p_t^{b_t}})$.

Since $\rho_{2b_0} : D_{2b_0-1} \rightarrow C_{2b_0}$ is an isomorphism, by Corollary [Appendix C.6](#) this kernel is $\oplus_{i=1}^t K_i$ where

$$K_i = D_{2b_0-1} \otimes P_{p_1^{b_1}} \otimes \cdots \otimes P_{p_{i-1}^{b_{i-1}}} \otimes \ker(\pi_{p_i^{b_i}}) \otimes Q_{p_{i+1}^{b_{i+1}}} \otimes \cdots \otimes Q_{p_t^{b_t}}$$

where Q_{p^a} is as described in Lemma [Appendix C.2](#).

A basis of D_{2b_0-1} is $\{x^i \mid 0 \leq i \leq 2^{b-1} - 1\}$. A basis of P_{p^b} is $\{x^i \mid 0 \leq i \leq p^b - 1\}$.

By Lemma [Appendix C.2](#), a basis for $\ker(\pi_{p^b})$ is $\{x^i \Phi_{p^b}(x) \mid i = 0, \dots, p^b - \phi(p^b) - 1 = p^{b-1} - 1\}$ and a basis for Q_{p^b} is $\{x^i \mid i = 0, \dots, \phi(p^b) - 1 = p^b - p^{b-1} - 1\}$.

[Theorem 11.10](#) now follows from Corollary [Appendix C.6](#). □

References

- [1] Fermi E, Pasta J, Ulam S. Studies of nonlinear problems. Technical Report. I, Los Alamos Scientific Laboratory Report No. LA-1940; 1955.
- [2] Dauxois T, Fermi, Pasta, Ulam, and a mysterious lady. *Phys Today* 2008;61(1):55–7.
- [3] Zabusky NJ, Kruskal MD. Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. *Phys Rev Lett* 1965;15:240–3. doi:10.1103/PhysRevLett.15.240.
- [4] Rink B. Proof of Nishida’s conjecture on anharmonic lattices. *Commun Math Phys* 2006;261(3):613–27.
- [5] Henrici A, Kappeler T. Results on normal forms for FPU chains. *Commun Math Phys* 2008;278(1):145–77.
- [6] Izrailev FM, Chirikov BV. Statistical properties of a nonlinear string. *Sov Phys Dokl* 1966;11:30–2.
- [7] Livi R, Pettini M, Ruffo S, Sparpaglione M, Vulpiani A. Equipartition threshold in nonlinear large Hamiltonian systems: the Fermi-Pasta-Ulam model. *Phys Rev A* 1985;31(2):1039.
- [8] Lvov YV, Onorato M. Double scaling in the relaxation time in the β -Fermi-Pasta-Ulam-Tsingou model. *Phys Rev Lett* 2018;120(14):144301.
- [9] Cretegnay T, Dauxois T, Ruffo S, Torcini A. Localization and equipartition of energy in the β -FPU chain: chaotic breathers. *Phys D* 1998;121(1):109–26.
- [10] Benettin G, Christodoulidi H, Ponno A. The Fermi-Pasta-Ulam problem and its underlying integrable dynamics. *J Stat Phys* 2013;152(2):195–212.
- [11] Benettin G, Ponno A. Time-scales to equipartition in the Fermi-Pasta-Ulam problem: finite-size effects and thermodynamic limit. *J Stat Phys* 2011;144(4):793–812.
- [12] Fucito F, Marchesoni F, Marinari E, Parisi G, Peliti L, Ruffo S, et al. Approach to equilibrium in a chain of nonlinear oscillators. *J Phys* 1982;43(5):707–13.
- [13] Onorato M, Vozella L, Proment D, Lvov YV. Route to thermalization in the α -Fermi-Pasta-Ulam system. *Proc Natl Acad Sci* 2015;112(14):4208–13.
- [14] Pistone L, Onorato M, Chibbaro S. Thermalization in the discrete nonlinear Klein-Gordon chain in the wave-turbulence framework. *EPL (Europhys Lett)* 2018;121(4):44003.
- [15] Falkovich G, Lvov V, Zakharov V. Kolmogorov spectra of turbulence. Springer, Berlin; 1992.
- [16] Rink B. Symmetry and resonance in periodic FPU chains. *Commun Math Phys* 2001;218(3):665–85.
- [17] Krasitskii V. On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J Fluid Mech* 1994;272:1–20.
- [18] Janssen P. On some consequences of the canonical transformation in the Hamiltonian theory of water waves. *J Fluid Mech* 2009;637(1):1–44.
- [19] Dyachenko A, Lvov Y, Zakharov V. Five-wave interaction on the surface of deep fluid. *Phys D* 1995;87(1–4):233–61.
- [20] Laatsch RG, et al. Extensions of subadditive functions. *Pacific J Math* 1964;14:209–15.
- [21] Bruckner A, et al. Minimal superadditive extensions of superadditive functions. *Pacific J Math* 1960;10:1155–62.
- [22] Matkowski J. Subadditive periodic functions. *Opuscula Math* 2011;31:75–96.
- [23] Wigert S. Sur l’ordre de grandeur du nombre des diviseurs d’un entier. *Almqvist & Wiksells*; 1907.
- [24] Giorgilli A, Paleari S, Penati T. Local chaotic behaviour in the Fermi-Pasta-Ulam system. *Discrete Contin Dyn Syst Ser B* 2005;5(4):991.
- [25] Paleari S, Penati T. Equipartition times in a Fermi-Pasta-Ulam system. *Discrete Contin Dyn Syst* 2005;710–19.
- [26] Harper KL, Bustamante MD, Nazarenko SV. Quadratic invariants for discrete clusters of weakly interacting waves. *J Phys A Math Theor* 2013;46(24):245501.
- [27] Lam TY, Leung KH. On vanishing sums of roots of unity. *J Algebra* 2000;224(1):91–109.