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Canonical Hamiltonians for waves in inhomogeneous media

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We obtain a canonical form of a quadratic Hamiltonian for linear waves in a weakly inhomogeneous medium. This is achieved by using the Wentzel–Kramers–Brillouin representation of wave packets. The canonical form of the Hamiltonian is obtained via the series of canonical Bogolyubov-type and near-identical transformations. Various examples of the application illustrating the main features of our approach are presented. The knowledge of the Hamiltonian structure for linear wave systems provides a basis for developing a theory of weakly nonlinear random waves in inhomogeneous media generalizing the theory of homogeneous wave turbulence. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3054275]

I. INTRODUCTION

In order to analyze the behavior of a general nonlinear system, the first necessary step is to analyze the linear system. The classical examples are finite-dimensional Hamiltonian systems with coupled degrees of freedom. The dynamical behavior of such systems for small excitations is determined by the form of the quadratic part of their Hamiltonians, which correspond to the linear dynamics. A detailed classification of the canonical normal forms for such Hamiltonians was given by Galin, and it was summarized in one of the appendices in the book of Arnold.[1] Another class of examples of nonlinear systems, which behavior crucially depends on the linear part, is weakly interacting dispersive waves in continuous media that are studied in wave turbulence (WT). These systems often have analogs among discrete systems, e.g., a chain of coupled oscillators, and correspondingly their quadratic Hamiltonians have analogs among Galin–Arnold canonical forms. Examples where WT has important applications are surface gravity waves,[2] β-plane turbulence in oceans and atmospheres,[3,4] internal waves in the ocean,[5] weak magnetohydrodynamics (MHD) turbulence,[6] Bose–Einstein condensate (BEC),[7,8] and plasmas.[9] Traditionally, WT theory is applied to homogeneous systems. The quadratic Hamiltonian describing linear interactions in such homogeneous systems is given by[10]

$$H_2 = \int \left(A_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k} + \frac{1}{2}(B_{\mathbf{k}} a_{\mathbf{k}} a_{-\mathbf{k}} + B^*_{\mathbf{k}} a^*_{\mathbf{k}} a^*_{-\mathbf{k}}) \right) d\mathbf{k}, \tag{1}$$

where $a_{\mathbf{k}}$ is a complex-valued field and the bold face denotes a d dimensional vector in $\mathbb{R}^d$. The linear canonical transformation to the new variables $b_{\mathbf{k}}$ that was first used by Bogolyubov in 1958 for the system of fermions brings this Hamiltonian to its normal form,

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$$H_2 = \int \omega_{\mathbf{k}} |b_{\mathbf{k}}|^2 d\mathbf{k}, \tag{2}$$

where $\omega_{\mathbf{k}}$ is a linear dispersive relationship. The formalism of WT significantly enhanced our understanding of spectral energy transfer in ocean, atmosphere, plasma, and other system of nonlinear waves.[10] WT deals with weakly nonlinear waves with quasirandom phases. Using WT, one can derive a kinetic equation for the wave spectrum, which evolves due to resonant wave interactions.

In order for the resonant energy transfers to occur, certain resonance conditions need to be satisfied. In particular, for the systems dominated by three wave interactions, such as internal waves in the ocean[5] or capillary waves,[11] these conditions are given by

$$\begin{aligned} \omega_{\mathbf{k}} &= \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \\ \mathbf{k} &= \mathbf{k}_1 + \mathbf{k}_2. \end{aligned} \tag{3}$$

Similar four-wave resonant conditions should be satisfied for the resonant interactions in the four-wave systems.

Often, WT is not spatially homogeneous and its statistical properties vary in space due to a trapping potential, inhomogeneous background density, or an inhomogeneous velocity field. Examples of such systems are the BEC in the presence of a trapping potential[8] and an interaction of the long "aged" gravity waves with a swell.[12] In general, the idea is to consider a small amplitude high frequency perturbation of a large-scale solution of the dynamic equation (e.g., the condensate). The effects of this coordinate dependent background solution can most easily be understood using a wave-packet formalism. This formalism was first used to approximate the Schrödinger's wave function by a quasimonochromatic wave by Wentzel,[13] Kramers,[14] and Brillouin.[15] Their initials give the term WKB approximation. The WKB approximation is applicable if the wave-packet wavelength l is much shorter than the characteristic wavelength L of a large-scale solution,

$$\epsilon = \frac{l}{L} \ll 1.$$

The essence of WKB approach is that the wave numbers, characterizing the wavepacket, are the functions of coordinates. This is due to the distortion of the wave packets by the media, leading to a spatial wavenumber dependence. Such spatial wave number dependence may have a dramatic effect on nonlinear resonant wave interactions. Indeed, the resonant conditions (3) may now be satisfied only in a finite part of a domain or, for a particular wave packet, only for a finite time.

The goal of this paper is to use the spatially dependent WKB wave packets to find a canonical form for the quadratic Hamiltonian for the inhomogeneous systems. This problem can be considered as an extension of certain (oscillatory) members from the Galin–Arnold classification of quadratic forms onto the infinite-dimensional and continuous space. The physical motivation for such a formulation is that, since the Hamiltonian description is natural for WT in homogeneous systems, it lays down a necessary framework for generalization onto the inhomogeneous media.

To begin, we write down the general Hamiltonian for the system of linear waves propagating in the inhomogeneous background. We will show in Sec. II that such general Hamiltonian for the variable $a_{\mathbf{q}}$ is given by the following quadratic form:

$$H = \int \left(A(\mathbf{q},\mathbf{q}_1) a_{\mathbf{q}} a^*_{\mathbf{q}_1} + \frac{1}{2}(B(\mathbf{q},\mathbf{q}_1) a_{\mathbf{q}} a_{-\mathbf{q}_1} + B^*(\mathbf{q},\mathbf{q}_1) a^*_{\mathbf{q}} a^*_{-\mathbf{q}_1}) \right) d\mathbf{q} d\mathbf{q}_1. \tag{4}$$

The main result of the present paper is that this general Hamiltonian (4) can be transformed to the following canonical form:

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$$H = \int c_{\mathbf{kx}}[\omega_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}} + i\{\omega_{\mathbf{kx}},\cdot\}]c^*_{\mathbf{kx}}d\mathbf{k}d\mathbf{x}. \tag{5}$$

Here $\omega_{\mathbf{kx}}$ and $c_{\mathbf{kx}}$ are the new position-dependent dispersion relationship and normal field variable, correspondingly, and a Poisson bracket is defined by

$$\{f,g\} = \nabla_{\mathbf{k}}f\cdot\nabla_{\mathbf{x}}g - \nabla_{\mathbf{k}}g\cdot\nabla_{\mathbf{x}}f.$$

This novel canonical Hamiltonian is a generalization of the Hamiltonian (2) for the inhomogeneous systems. The approach used in the paper can be viewed as a generalization of the Bogolyubov transformation, which diagonalizes Hamiltonian (1) to the form (2) via canonical transformation. Similarly, Hamiltonian (4) can be transformed into the canonical form (5), however, now using near-canonical transformations. In the Hamiltonian (5), the second and the third term in the brackets correspond to the higher order corrections to the dispersion relation due to the inhomogeneity. We prove that the Hamiltonian (4) can be transformed into a canonical form (5) in the case when the heterogeneity is weak. Formally, the requirement of weak inhomogeneity means that the coefficients $A(\mathbf{q},\mathbf{q}_1)$ and $B(\mathbf{q},\mathbf{q}_1)$ are strongly peaked at $\mathbf{q}-\mathbf{q}_1=0$, i.e., $A(\mathbf{q},\mathbf{q}_1)=0$ for $|\mathbf{q}-\mathbf{q}_1|>\varepsilon$ for some small ε. Based on this requirement, we derive below the renormalized dispersion relationship and the transformation formulas from $a_{\mathbf{k}}$ to $c_{\mathbf{kx}}$ accurate up to the first order in ε. It turns out that just Bogolyubov's transformation is not enough in this case. The phase coordinate systems should also be perturbed by a near-identity transformation in addition to the Bogolyubov's rotation. Then, the Hamiltonian becomes diagonal up to the first order in $\boldsymbol{\epsilon}$.

From the novel canonical form of the Hamiltonian given by Eq. (5), the traditional radiative action balance equation can easily be obtained,

$$\frac{\partial n_{\mathbf{kx}}}{\partial t} + \nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\nabla_{\mathbf{x}}n_{\mathbf{kx}} - \nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\nabla_{\mathbf{k}}n_{\mathbf{kx}} = 0,$$

or, shorter,

$$\frac{\partial n_{\mathbf{kx}}}{\partial t} + \{\omega_{\mathbf{kx}}, n_{\mathbf{kx}}\} = 0, \tag{6}$$

where $n_{\mathbf{kx}}$ denotes the ensemble average of the squared amplitude of the wave, i.e., $n_{\mathbf{kx}} \equiv \langle|c_{\mathbf{kx}}|^2\rangle$. Equation (6) is now a standard equation which is used in statistical modeling of wave systems.[16] It is a wave analog of the Liouville's theorem or the continuity equations for distribution function of statistical mechanics.[17,18] In wave systems there are wave quasiparticles instead of particles, following different rays instead of individual particle trajectories. Wave action distribution function moves in the multidimensional wavenumber-coordinate phase space.

The paper is organized as follows. In Sec. II, we give simple and instructive examples that motivate the study of inhomogeneous WKB systems. In Sec. III, we introduce the window transforms and other formulas that will be extensively used later. In Sec. IV, we discuss the case of a nearly diagonal Hamiltonian. We show how it can be transformed to a canonical form (2) and provide a couple of representative examples. In Sec. V, we study the Hamiltonian in a general form (4). We present the series of near-canonical and near-identical transformations that bring the Hamiltonian (4) to the form (5). We also demonstrate the application of this approach to the nonlinear Schrödinger equation with the condensate.

II. MOTIVATION

In this section, we motivate our study of inhomogeneous systems by considering an example of the interaction of short waves on the background of a long wave. We describe the cases of both three-wave and four-wave Hamiltonian systems.

Three-wave case. The quadratic Hamiltonian of a three-wave system with small-scale pertur-

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bations on the background of the large-scale excitations is given by Eq. (4). In order to show that, we start with a standard three-wave Hamiltonian,[10]

$$H_3 = \int \Omega_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k} + \frac{1}{2}\int (V^{\mathbf{k}}_{\mathbf{lm}} a^*_{\mathbf{k}} a_{\mathbf{l}} a_{\mathbf{m}} + \text{c.c.}) \delta^{\mathbf{k}}_{\mathbf{lm}} d\mathbf{k} d\mathbf{l} d\mathbf{m}, \tag{7}$$

where V is an interaction coefficient. Then, the equations of motion for the variable $a_{\mathbf{k}}$ assume a standard form,

$$i\dot{a}_{\mathbf{k}} = \frac{\delta H_3}{\delta a^*_k} = \Omega_{\mathbf{k}} a_{\mathbf{k}} + \int \left(\frac{1}{2} V^{\mathbf{k}}_{\mathbf{lm}} a_{\mathbf{l}} a_{\mathbf{m}} \delta^{\mathbf{k}}_{\mathbf{lm}} + (V^{\mathbf{l}}_{\mathbf{km}})^* a_{\mathbf{l}} a^*_{\mathbf{m}} \delta^{\mathbf{l}}_{\mathbf{km}} \right) d\mathbf{l} d\mathbf{m}. \tag{8}$$

Suppose that a large-scale solution of (8) is given by $C_{\mathbf{k}}$. We consider a perturbed solution $a_{\mathbf{k}} = C_{\mathbf{k}} + c_{\mathbf{k}}$, where $c_{\mathbf{k}}$ is a small-scale perturbation of C_k. Equation of motion for $c_{\mathbf{k}}$ attains the following form:

$$i\dot{c}_{\mathbf{k}} = \Omega_{\mathbf{k}} c_{\mathbf{k}} + \int \left[\frac{1}{2} V^{\mathbf{k}}_{\mathbf{lm}} (C_{\mathbf{l}} c_{\mathbf{m}} + C_{\mathbf{m}} c_{\mathbf{l}} + c_{\mathbf{l}} c_{\mathbf{m}}) \delta^{\mathbf{k}}_{\mathbf{lm}} + (V^{\mathbf{l}}_{\mathbf{km}})^* (C_{\mathbf{l}} c^*_{\mathbf{m}} + C^*_{\mathbf{m}} c_{\mathbf{l}} + c_{\mathbf{l}} c^*_{\mathbf{m}}) \delta^{\mathbf{l}}_{\mathbf{km}} \right] d\mathbf{l} d\mathbf{m}.$$

Now, we use the fact that $C_{\mathbf{k}}$ is a known exact solution for Eq. (8) to obtain

$$i\dot{c}_{\mathbf{k}} = \int A(\mathbf{k},\mathbf{l}) c_{\mathbf{l}} d\mathbf{l} + \frac{1}{2}\int B(\mathbf{k},\mathbf{l}) c^*_{-\mathbf{l}} d\mathbf{l} + \int \left[\frac{1}{2} V^{\mathbf{k}}_{\mathbf{lm}} c_{\mathbf{l}} c_{\mathbf{m}} \delta^{\mathbf{k}}_{\mathbf{lm}} + (V^{\mathbf{l}}_{\mathbf{km}})^* c_{\mathbf{l}} c^*_{\mathbf{m}} \delta^{\mathbf{l}}_{\mathbf{km}} \right] d\mathbf{l} d\mathbf{m}, \tag{9}$$

where

$$\begin{aligned} A(\mathbf{k},\mathbf{l}) &= \Omega_{\mathbf{l}} \delta^{\mathbf{k}}_{\mathbf{l}} + V^{\mathbf{k}}_{\mathbf{l},\mathbf{k}-\mathbf{l}} C_{\mathbf{k}-\mathbf{l}} + (V^{\mathbf{l}}_{\mathbf{k},\mathbf{l}-\mathbf{k}})^* C^*_{\mathbf{l}-\mathbf{k}}, \\ B(\mathbf{k},\mathbf{l}) &= 2V^{\mathbf{k}-\mathbf{l}}_{\mathbf{k},-\mathbf{l}} C_{\mathbf{k}-\mathbf{l}}. \end{aligned} \tag{10}$$

Equation (9) corresponds to the following Hamiltonian:

$$H = \int A(\mathbf{k},\mathbf{l}) c_{\mathbf{l}} c^*_{\mathbf{k}} d\mathbf{l} d\mathbf{k} + \frac{1}{2}\int [B(\mathbf{k},\mathbf{l}) c^*_{-\mathbf{l}} c^*_{\mathbf{k}} + \text{c.c.}] d\mathbf{k} d\mathbf{l} + \frac{1}{2}\int [V^{\mathbf{k}}_{\mathbf{lm}} c^*_{\mathbf{k}} c_{\mathbf{l}} c_{\mathbf{m}} \delta^{\mathbf{k}}_{\mathbf{lm}} + \text{c.c.}] d\mathbf{k} d\mathbf{l} d\mathbf{m}. \tag{11}$$

This appears to be a standard form of the Hamiltonian for the wave system dominated by three wave interactions in the inhomogeneous media. Quadratic in $c_{\mathbf{k}}$ part of this Hamiltonian is given by (4), while cubic part of this Hamiltonian is a standard three-wave interaction Hamiltonian.

Four-wave case. Four-wave systems are similar to three-wave systems when small-scale perturbations on the background of the large-scale excitations are considered. Indeed, we show below that the quadratic part of a four-wave Hamiltonian of small-scale perturbation has the form (4). We start from a standard four-wave Hamiltonian,[10]

$$H_4 = \int \Omega_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k} + \frac{1}{2}\int T^{\mathbf{kl}}_{\mathbf{ms}} a^*_{\mathbf{k}} a^*_{\mathbf{l}} a_{\mathbf{m}} a_{\mathbf{s}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{k} d\mathbf{l} d\mathbf{m} d\mathbf{s}, \tag{12}$$

where T is an interaction coefficient. The corresponding equation of motion takes the form

$$i\dot{a}_{\mathbf{k}} = \Omega_{\mathbf{k}} a_{\mathbf{k}} + \int T^{\mathbf{kl}}_{\mathbf{ms}} a^*_{\mathbf{l}} a_{\mathbf{m}} a_{\mathbf{s}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m} d\mathbf{s}. \tag{13}$$

We then consider a perturbed solution $a_{\mathbf{k}} = C_{\mathbf{k}} + c_{\mathbf{k}}$, where $c_{\mathbf{k}}$ is a small-scale perturbation. Assuming that $C_{\mathbf{k}}$ is an exact solution to the equation of motion with Hamiltonian (13), we obtain the following equation of motion for $c_{\mathbf{k}}$:

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$$i\dot{c}_{\mathbf{k}} = \Omega_{\mathbf{k}} c_{\mathbf{k}} + \int T^{\mathbf{kl}}_{\mathbf{ms}} (2C^*_{\mathbf{l}} C_{\mathbf{m}} c_{\mathbf{s}} + c^*_{\mathbf{l}} C_{\mathbf{m}} C_{\mathbf{s}}) \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m} d\mathbf{s} + \int T^{\mathbf{kl}}_{\mathbf{ms}} (C^*_{\mathbf{l}} c_{\mathbf{m}} c_{\mathbf{s}} + 2c^*_{\mathbf{l}} c_{\mathbf{m}} C_{\mathbf{s}}) \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m} d\mathbf{s}$$
$$+ \int T^{\mathbf{kl}}_{\mathbf{ms}} c^*_{\mathbf{l}} c_{\mathbf{m}} c_{\mathbf{s}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m} d\mathbf{s}.$$

Since $C_{\mathbf{k}}$ is a known large-scale solution, we obtain

$$i\dot{c}_{\mathbf{k}} = \int A(\mathbf{k},\mathbf{s}) c_{\mathbf{s}} d\mathbf{s} + \frac{1}{2}\int B(\mathbf{k},\mathbf{s}) c^*_{-\mathbf{s}} d\mathbf{s} + \int \left[\frac{1}{2} W^{\mathbf{k}}_{\mathbf{lm}} c_{\mathbf{l}} c_{\mathbf{m}} + (W^{\mathbf{l}}_{\mathbf{km}})^* c_{\mathbf{l}} c^*_{\mathbf{m}}\right] d\mathbf{l} d\mathbf{m}$$
$$+ \int T^{\mathbf{kl}}_{\mathbf{ms}} c^*_{\mathbf{l}} c_{\mathbf{m}} c_{\mathbf{s}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m} d\mathbf{s}, \tag{14}$$

where we defined the kernels $A(\mathbf{k},\mathbf{s})$ and $B(\mathbf{k},\mathbf{s})$ as

$$A(\mathbf{k},\mathbf{s}) = \Omega_{\mathbf{s}} \delta^{\mathbf{k}}_{\mathbf{s}} + 2\int T^{\mathbf{kl}}_{\mathbf{ms}} C^*_{\mathbf{l}} C_{\mathbf{m}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{l} d\mathbf{m},$$
$$B(\mathbf{k},\mathbf{s}) = 2\int T^{\mathbf{k},-\mathbf{s}}_{\mathbf{m},\mathbf{l}} C_{\mathbf{m}} C_{\mathbf{l}} \delta^{\mathbf{k},-\mathbf{s}}_{\mathbf{m},\mathbf{l}} d\mathbf{l} d\mathbf{m}, \tag{15}$$

and

$$W^{\mathbf{k}}_{\mathbf{lm}} = 2\int T^{\mathbf{ks}}_{\mathbf{ml}} C^*_{\mathbf{s}} \delta^{\mathbf{ks}}_{\mathbf{lm}} d\mathbf{s}.$$

The linear part of Eq. (14) has the same form as the linear part of the corresponding equation obtained for the three-wave case (9). Thus, this linear part corresponds to the same first two terms as in Hamiltonian (11). Note also similarity of the quadratic terms in (9) and (14). Note that (14) corresponds to the following Hamiltonian:

$$H = \int A(\mathbf{k},\mathbf{l}) c_{\mathbf{l}} c^*_{\mathbf{k}} d\mathbf{l} d\mathbf{k} + \frac{1}{2}\int [B(\mathbf{k},\mathbf{l}) c^*_{-\mathbf{l}} c^*_{\mathbf{k}} + \text{c.c.}] d\mathbf{k} d\mathbf{l} + \frac{1}{2}\int [W^{\mathbf{k}}_{\mathbf{lm}} c^*_{\mathbf{k}} c_{\mathbf{l}} c_{\mathbf{m}} + \text{c.c.}] d\mathbf{k} d\mathbf{l} d\mathbf{m}$$
$$+ \frac{1}{2}\int T^{\mathbf{kl}}_{\mathbf{ms}} c^*_{\mathbf{k}} c^*_{\mathbf{l}} c_{\mathbf{m}} c_{\mathbf{s}} \delta^{\mathbf{kl}}_{\mathbf{ms}} d\mathbf{k} d\mathbf{l} d\mathbf{m} d\mathbf{s}. \tag{16}$$

This appears to be a standard Hamiltonian for the wave system with four-wave interactions in the presence of spatial inhomogeneity. Indeed, the quadratic part (first line) of this Hamiltonian is the Hamiltonian (4). The cubic term is the three-wave interactions with the background large-scale wave (i.e., four-wave interaction where the role of the fourth wave is assumed by the background wave). Notice that unlike traditional three-wave interactions in a homogeneous environment, momentum is *not* conserved by this term. This is the effect of breaking of spatial symmetry by an inhomogeneous background. Lastly, the quartic term (third line) is the standard four-wave interaction Hamiltonian. We show in this paper that the quadratic part of this Hamiltonian may be reduced to the novel canonical Hamiltonian for spatially inhomogeneous systems (5).

In this section we have demonstrated that if the general wave system is dominated by three-wave or four-wave interactions and consists of short scale waves superimposed on known large-scale motion, its quadratic Hamiltonian is given by Eq. (4).

III. PRELIMINARIES

In this section, we set up the stage for formulation of our results. Here, we give basic definitions and obtain frequently used formulas.

We use the following definition of direct and inverse Fourier transforms:

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$$\hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d}\int g(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x},$$

$$g(\mathbf{x}) = \int \hat{g}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}.$$

Next, we generalize the Fourier transform to spatially inhomogeneous systems. In order to do that, we use a window transform of $g(\mathbf{x})$,

$$\Gamma[g(\mathbf{x})] \equiv \tilde{g}(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int f(\varepsilon^*|\mathbf{x}-\mathbf{x}_0|)g(\mathbf{x}_0)e^{-i\mathbf{k}\cdot\mathbf{x}_0}d\mathbf{x}_0. \tag{17}$$

Here, $f(x)$ is an arbitrary fast decaying at infinity window function. The parameter ε^* is defined by the spatial scales of the inhomogeneity and the propagating wave packets in the following manner. First, we introduce the characteristic length of inhomogeneity to be of the order $1/\varepsilon$. Then, we take the width of the window, which is of the order $1/\varepsilon^*$, to be much smaller than the characteristic length of inhomogeneity. On the other hand, the width of the window is chosen to be much larger than the wavelength of the waves that propagate in the inhomogeneous medium, which is of the order of 1. Therefore, we have

$$\varepsilon \ll \varepsilon^* \ll 1. \tag{18}$$

The special case when $f(x)=\exp(-x^2)$ is called Gabor transform.[19] Note that when ε^* approaches zero, $f(\varepsilon^* x)$ approaches the constant function with the value one. Consequently, the Gabor transform becomes a Fourier transform. Therefore the Fourier transform can be seen as an averaging over an infinitely large window.

The inverse of the window transform (17) is given by

$$g(\mathbf{x}) = \int \tilde{g}(\mathbf{x},\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \tag{19}$$

where we have used $f(0)=1$. We emphasize that Eq. (19) and all the formulas that we obtain below can be obtained using any fast decaying at infinity window function and are independent of the particular form of $f(x)$ as long as it is sufficiently smooth.

Now, we present the formulas for the window transform, which will be useful later. First, we express the window transform $\tilde{g}(\mathbf{x},\mathbf{k})$ in terms of the Fourier transform $\hat{g}(\mathbf{k})$,

$$\tilde{g}(\mathbf{x},\mathbf{k}) = \frac{1}{(\varepsilon^*)^d}\int \hat{f}(|\mathbf{k}-\mathbf{q}|/\varepsilon^*)e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{x}}\hat{g}(\mathbf{q})d\mathbf{q}. \tag{20}$$

Next, we express the Fourier image $\hat{g}(\mathbf{k})$ in terms of the window variable $\tilde{g}(\mathbf{x},\mathbf{k})$,

$$\hat{g}(\mathbf{k}) = \left(\frac{\varepsilon^*}{\sqrt{\pi}}\right)^d\int \tilde{g}(\mathbf{x},\mathbf{k})d\mathbf{x}. \tag{21}$$

By combining Eqs. (20) and (21), we obtain the following formula:

$$\tilde{g}(\mathbf{x},\mathbf{k}) = \left(\frac{1}{\sqrt{\pi}}\right)^d\int \hat{f}(|\mathbf{q}-\mathbf{k}|/\varepsilon^*)e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{x}}\tilde{g}(\mathbf{x}',\mathbf{q})d\mathbf{q}d\mathbf{x}'. \tag{22}$$

After introducing notations and formulas that will be extensively used below, we proceed to the discussion of the main results of the paper.

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IV. THE CASE OF NEARLY DIAGONAL HAMILTONIANS

A. Formulation and proof of the lemma

Let us start with Hamiltonian (4) without off-diagonal terms, so that $B(\mathbf{q},\mathbf{q}_1)\equiv 0$. This is a typical Hamiltonian for linear waves in weakly inhomogeneous media[10] expressed in terms of Fourier amplitudes $a_{\mathbf{q}}$ and $a^*_{\mathbf{q}_1}$ as

$$H=\int \Omega(\mathbf{q},\mathbf{q}_1)a_{\mathbf{q}}a^*_{\mathbf{q}_1}d\mathbf{q}d\mathbf{q}_1, \tag{23}$$

with a Hermitian kernel $\Omega(\mathbf{q}_1,\mathbf{q})=\Omega^*(\mathbf{q},\mathbf{q}_1)$, which is strongly peaked at $\mathbf{q}-\mathbf{q}_1=0$ (Ω has a finite support around $\mathbf{q}\simeq\mathbf{q}_1$). Therefore, we subsequently assume that there is a small parameter ε for which

$$\Omega(\mathbf{q}_1-\mathbf{q})=0, \tag{24}$$

when $|\mathbf{q}-\mathbf{q}_1|>\varepsilon$. A particular choice

$$\Omega(\mathbf{q}_1,\mathbf{q})=\omega(\mathbf{q}_1)\delta(\mathbf{q}-\mathbf{q}_1),$$

leads to the familiar form of the Hamiltonian (2).

The equation of motion for $a_{\mathbf{k}}$ is

$$i\dot{a}_{\mathbf{k}}=\frac{\delta H}{\delta a^*_{\mathbf{k}}}=\int \Omega_{\mathbf{qk}}a_{\mathbf{q}}d\mathbf{q}. \tag{25}$$

Lemma: *Consider the Hamiltonian (23) with $\Omega(\boldsymbol{q},\boldsymbol{q}_1)$ being a peaked function of $(\boldsymbol{q}-\boldsymbol{q}_1)$ [satisfying (24)] and a smooth function of $(\boldsymbol{q}+\boldsymbol{q}_1)$. Then there exist a near-canonical change in variables $a_{\boldsymbol{k}}\to\check{a}_{\boldsymbol{kx}}$ such that in the new variables the equation of motion can be written in the Hamiltonian form,*

$$i\frac{\partial}{\partial t}\check{a}_{\boldsymbol{kx}}=\frac{\delta H_f}{\delta\check{a}^*_{\boldsymbol{kx}}},$$

with the filtered Hamiltonian in a canonical form,

$$H_f=\int \check{a}_{\boldsymbol{kx}}[\omega_{\boldsymbol{kx}}-\boldsymbol{x}\cdot\nabla_{\boldsymbol{x}}\omega_{\boldsymbol{kx}}+i\{\omega_{\boldsymbol{kx}},\cdot\}]\check{a}^*_{\boldsymbol{kx}}d\boldsymbol{k}d\boldsymbol{x}, \tag{26}$$

where $\omega_{\boldsymbol{kx}}$ is the position-dependent frequency related to $\Omega(\boldsymbol{q},\boldsymbol{q}_1)$ via the Wigner transform

$$\omega_{\boldsymbol{kx}}=\int e^{i\boldsymbol{m}\cdot\boldsymbol{x}}\Omega(\boldsymbol{k}-\boldsymbol{m}/2,\boldsymbol{k}+\boldsymbol{m}/2)d\boldsymbol{m}. \tag{27}$$

Proof: In order to obtain the new variables $\check{a}_{\mathbf{kx}}$, we first make a window transform via (17), which is then followed by a near-identical transformation. The idea of the proof is to use the peakness of the kernel $\Omega(\mathbf{q}_1,\mathbf{q})$. We make a Taylor expansion around the peak, and then by neglecting the higher order terms we obtain the desired result.

We make a window transform from $a_{\mathbf{k}}$ to $\tilde{a}_{\mathbf{kx}}$ using Eq. (20). Differentiating Eq. (20) with respect to time, using the Eq. (25), and applying the inverse formula (21) yield

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$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \left(\frac{1}{\varepsilon^*}\right)^d \int \hat{f}(|\mathbf{k}-\mathbf{q}_1|/\varepsilon^*) e^{i(\mathbf{q}_1-\mathbf{k})\cdot\mathbf{x}} \Omega_{\mathbf{qq}_1} a_{\mathbf{q}} d\mathbf{q} d\mathbf{q}_1$$
$$= \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}(|\mathbf{k}-\mathbf{q}_1|/\varepsilon^*) e^{i(\mathbf{q}_1-\mathbf{k})\cdot\mathbf{x}} \Omega_{\mathbf{qq}_1} \tilde{a}_{\mathbf{qx}_1} d\mathbf{q} d\mathbf{q}_1 d\mathbf{x}_1. \tag{28}$$

Let us change variables from $(\mathbf{q},\mathbf{q}_1)$ to $(\mathbf{p},\mathbf{m})$ as

$$\mathbf{q} = \mathbf{p} - \mathbf{m}/2, \tag{29}$$

$$\mathbf{q}_1 = \mathbf{p} + \mathbf{m}/2. \tag{30}$$

Below it will be convenient to use

$$F(\mathbf{p},\mathbf{m}) \equiv \Omega(\mathbf{p}-\mathbf{m}/2, \mathbf{p}+\mathbf{m}/2). \tag{31}$$

Next, we will approximate the right-hand side of Eq. (28) by a variation in some filtered Hamiltonian H_f, i.e., by $\delta H_f/\delta \tilde{a}^*_{\mathbf{kx}}$. We can rewrite Eq. (28) as

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}(|\mathbf{k}-\mathbf{p}-\mathbf{m}/2|/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}/2-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{p},\mathbf{m}) a_{\mathbf{p}-\mathbf{m}/2,\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1. \tag{32}$$

Let us make another change in variables $\mathbf{p} \to \mathbf{p}+\mathbf{m}/2$,

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}(|\mathbf{k}-\mathbf{p}-\mathbf{m}|/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{p}+\mathbf{m}/2,\mathbf{m}) a_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1. \tag{33}$$

In order to simplify Eq. (33), we are going to use the fact that $\Omega_{\mathbf{qq}_1}$ and $\hat{f}(\mathbf{k})$ are peaked functions of $(\mathbf{q}_1-\mathbf{q})$ and $\mathbf{k}$, respectively, and fast decaying at infinity. We also keep only first order terms in spatial derivatives, neglecting second and higher order terms. Then, we could write

$$\hat{f}(|\mathbf{k}-\mathbf{p}-\mathbf{m}|/\varepsilon^*) = \hat{f}(|\mathbf{k}-\mathbf{p}|/\varepsilon^*) + \mathbf{m}\cdot\nabla_{\mathbf{p}}\hat{f}(|\mathbf{k}-\mathbf{p}|/\varepsilon^*) + \text{h.o.t.} \tag{34}$$

Similarly, we obtain

$$F(\mathbf{p}+\mathbf{m}/2,\mathbf{m}) = F(\mathbf{k}+\mathbf{p}-\mathbf{k}+\mathbf{m}/2,\mathbf{m}) = F(\mathbf{k},\mathbf{m}) + (\mathbf{p}-\mathbf{k}+\mathbf{m}/2)\cdot\nabla_{\mathbf{k}}F(\mathbf{k},\mathbf{m}) + \text{h.o.t.}, \tag{35}$$

where h.o.t. denotes higher order terms. Now, we substitute the expansions (34) and (35) into Eq. (32), and after ignoring higher order terms, we obtain

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \left(\frac{1}{\sqrt{\pi}}\right)^d \int (\hat{f}(|\mathbf{k}-\mathbf{p}|/\varepsilon^*) + \mathbf{m}\cdot\nabla_{\mathbf{p}}\hat{f}(|\mathbf{k}-\mathbf{p}|/\varepsilon^*)) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}}$$
$$\times (F(\mathbf{k},\mathbf{m}) + (\mathbf{p}-\mathbf{k}+\mathbf{m}/2)\cdot\nabla_{\mathbf{k}}F(\mathbf{k},\mathbf{m}))\tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1. \tag{36}$$

Note that here we have an expansion with two different small parameters ε and ε^*, which obey Eq. (18).

In Appendix A, we show how to simplify the right-hand side of Eq. (36). As a result of this simplification, we obtain

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} + i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\tilde{a}_{\mathbf{kx}} - i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} + \underline{i(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}(\mathbf{x}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}})}$$
$$+ i/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} + \underline{(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\tilde{a}_{\mathbf{kx}}}. \tag{37}$$

Here we have underlined the terms that have two spacial derivatives. In the spirit of WKB

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approximation these terms will later be neglected. This equation can be written in the Hamiltonian form,

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \frac{\delta H_f}{\delta \tilde{a}^*_{\mathbf{kx}}},$$

where the filtered Hamiltonian takes the form

$$H_f = \int ((\omega_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}})|\tilde{a}_{\mathbf{kx}}|^2 + (\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}^*_{\mathbf{kx}}(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\tilde{a}_{\mathbf{kx}} + i\tilde{a}^*_{\mathbf{kx}}(\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\tilde{a}_{\mathbf{kx}}$$
$$- \nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} + (\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}(\mathbf{x}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}}) + 1/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}}))d\mathbf{k}d\mathbf{x}. \tag{38}$$

Now, we will use the general method of the WKB approximation. We will only keep the terms, which are of the first order in a small parameter ε. In our case, the small parameter ε characterizes the rate of spatial change in the position-dependent frequency $\omega_{\mathbf{kx}}$ and the dynamical variable $\tilde{a}_{\mathbf{kx}}$. To apply the WKB approximation to Eq. (37), we neglect the terms that have two derivatives with respect to $\mathbf{x}$ (underlined) because each spatial derivative is of the order ε small. As a result, we obtain the following equation of motion:

$$i\frac{\partial}{\partial t}\tilde{a}_{\mathbf{kx}} = \omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} + i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\tilde{a}_{\mathbf{kx}} - i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} + i/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}}. \tag{39}$$

However, Eq. (39) becomes non-Hamiltonian. Indeed, the corresponding functional,

$$H_f = \int ((\omega_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}})|\tilde{a}_{\mathbf{kx}}|^2 + i\tilde{a}^*_{\mathbf{kx}}(\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\tilde{a}_{\mathbf{kx}} - \nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} + 1/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}}))d\mathbf{k}d\mathbf{x}, \tag{40}$$

is not self-adjoint if $(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}} \neq 0$. Therefore, in order to obtain canonical equations of motion, another near-canonical change in variables needs to be performed,

$$\tilde{a}_{\mathbf{kx}}(t) = s_{\mathbf{kx}}\breve{a}_{\mathbf{kx}}(t), \tag{41}$$

where $s_{\mathbf{kx}}$ is some time-independent function to be determined below. Note that transformation (41) is canonical if and only if

$$|s_{\mathbf{kx}}|^2 = 1.$$

Therefore, we need to find such $s_{\mathbf{kx}}$ that the system becomes Hamiltonian in terms of new variables $\breve{a}_{\mathbf{kx}}$ and the transformation (41) is near canonical, i.e., $|s_{\mathbf{kx}}| \approx 1$. We substitute Eq. (41) into Eq. (39) to obtain

$$is_{\mathbf{kx}}\frac{\partial}{\partial t}\breve{a}_{\mathbf{kx}} = s_{\mathbf{kx}}[(\omega_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_x\omega_{\mathbf{kx}})\breve{a}_{\mathbf{kx}} + i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\breve{a}_{\mathbf{kx}} - i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\breve{a}_{\mathbf{kx}}]$$
$$+ \breve{a}_{\mathbf{kx}}[i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}s_{\mathbf{kx}} - i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}s_{\mathbf{kx}} + i/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}s_{\mathbf{kx}}].$$

If we find $s_{\mathbf{kx}}$ that satisfies the equation

$$\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}s_{\mathbf{kx}} - \nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}s_{\mathbf{kx}} = \tfrac{1}{2}s_{\mathbf{kx}}(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}, \tag{42}$$

then the equation of motion in the new variables $\breve{a}_{\mathbf{kx}}$ takes the canonical form

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$$i\frac{\partial}{\partial t}\breve{a}_{\mathbf{kx}} = (\omega_{\mathbf{kx}} - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}})\breve{a}_{\mathbf{kx}} - i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\breve{a}_{\mathbf{kx}} + i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\breve{a}_{\mathbf{kx}}, \tag{43}$$

with the corresponding Hamiltonian (26). In order to find a solution of Eq. (42), we make a change in variables,

$$g_{\mathbf{kx}} = 2\ln s_{\mathbf{kx}}, \tag{44}$$

to obtain

$$\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}g_{\mathbf{kx}} - \nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}g_{\mathbf{kx}} = (\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}. \tag{45}$$

We find the solution of Eq. (45) using the method of characteristics. The characteristics $(\mathbf{x}(\tau),\mathbf{k}(\tau))$ are given by the following equations:

$$\frac{d\mathbf{x}}{d\tau} = \nabla_{\mathbf{k}}\omega_{\mathbf{kx}},$$
$$\frac{d\mathbf{k}}{d\tau} = -\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}, \tag{46}$$

where τ is a parameter along the characteristics. Physically, these characteristics correspond to the trajectories (rays) of WKB wavepackets in the $(\mathbf{x},\mathbf{k})$ space. The solution of Eq. (45) is given by

$$g(\mathbf{x}(\tau),\mathbf{k}(\tau)) = \int_0^\tau (\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}d\tau'.$$

Now, we use Eq. (44) and then Eq. (41) in order to obtain the new variable $\breve{a}_{\mathbf{kx}}$ out of the Gabor variable $\tilde{a}_{\mathbf{kx}}$. The dynamics of the new variable $\breve{a}_{\mathbf{kx}}$ is described by the filtered Hamiltonian (26). □

Note that for the special case $(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}=0$, the Gabor variables $\tilde{a}_{\mathbf{kx}}$ provide a Hamiltonian structure (26). Then, in this special case, there is no need in performing the second transformation (41). We will see below in the examples that this observation may significantly simplify the applications of the lemma.

To describe the statistical properties of spectral energy transfer in the systems, it is convenient to define a position-dependent wave action as

$$\breve{n}_{\mathbf{kx}} \equiv \langle|\breve{a}_{\mathbf{kx}}|^2\rangle. \tag{47}$$

Using this definition, one obtains from (26) the familiar form (6) of the kinetic equation (sometimes called radiative balance equation) in a weakly inhomogeneous media, by using Eq. (43). The resulting equation for the time evolution of $\breve{n}_{\mathbf{kx}}$ is Eq. (6) in Sec. I. This is the so-called waveaction transport which is typical for WKB systems.

B. Relation to the Wigner transformation

One can also derive waveaction transport equation (6) directly from equation of motion (25), without obtaining first the Hamiltonian structure (5). To do this, we define the Wigner waveaction by using the Wigner transformation,

$$n^{\mathrm{W}}_{\mathbf{kx}} \equiv \int e^{i\mathbf{m}\cdot\mathbf{x}}\langle\hat{a}_{\mathbf{k}+\frac{1}{2}\mathbf{m}}\hat{a}^*_{\mathbf{k}-\frac{1}{2}\mathbf{m}}\rangle d\mathbf{m}. \tag{48}$$

Then the Wigner waveaction $n^{\mathrm{W}}_{\mathbf{kx}}$ obeys the same kinetic equation (6). The proof can be found, for example, in Ref. 20. The idea of the proof is to calculate the time evolution of waveaction $n^{\mathrm{W}}_{\mathbf{kx}}$ by using definition (48) and equation of motion (25). Then one uses transformation similar to (29)

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and (30) and expands $n^{\mathrm{W}}_{\mathbf{kx}}$ using the smallness of $\mathbf{m}$, and finally uses integration by parts to obtain (6)—see Ref. 20 for details.

To address the question on how waveaction $n^{\mathrm{W}}_{\mathbf{kx}}$, defined through the Wigner transformation, is related to the wave action $\check{n}_{\mathbf{kx}}$, defined through the Gabor variables, we substitute (21) into the definition of (48) to write

$$n^{\mathrm{W}}_{\mathbf{kx}} = \left(\frac{\varepsilon^*}{\sqrt{\pi}}\right)^{2d} \int \langle \check{a}_{\mathbf{k}+\mathbf{m}/2,\mathbf{x}'} \check{a}^*_{\mathbf{k}-\mathbf{m}/2,x''} \rangle e^{i\mathbf{m}\cdot\mathbf{x}} d\mathbf{x}' d\mathbf{x}'' d\mathbf{m}.$$

Taking into account that $\check{a}_{\mathbf{k}+\mathbf{m}/2,\mathbf{x}'}$ and $\check{a}^*_{\mathbf{k}-\mathbf{m}/2,x''}$ are slow functions of $\mathbf{x}'$ and $\mathbf{x}''$, one can neglect this slow coordinate dependence relative to fast coordinate dependence in the exponent. This allows to perform $\mathbf{x}'$ and $\mathbf{x}''$ integrations to obtain a delta function with respect to the $\mathbf{m}$ argument. Consequently, we obtain that these two waveactions, (48) and (47) are approximately proportional to each other,

$$n^{\mathrm{W}}_{\mathbf{k},\mathbf{x}} \propto \check{n}_{\mathbf{k},\mathbf{x}}.$$

There are several important advantages of our method. First, it allows to rigorously write the equation of motion for the field variable $a_{\mathbf{k},\mathbf{x}}$ in addition to the transfer equation of (6). In addition, our approach shows how to derive the kinetic equation (6) to a much broader class of nonlinear systems, those described by Eq. (4) with nonzero value of $B(\mathbf{q},\mathbf{q}_1) \neq 0$, as we show in Sec. V. Lastly, Hamiltonian formulation helps to rigorously establish a wave turbulence theory and to take into account nonlinear wave interactions.

C. Example: Linear Schrödinger equation

Consider a one-dimensional example of a linear Schrödinger equation with a slowly varying potential. This equation is also referred to as linearized Gross–Pitaevsky equation. It is used to describe a formation of the BEC. It is given by

$$i\dot{\psi} = -\nabla_x^2 \psi + U(x)\psi. \tag{49}$$

This equation can be written in a Hamiltonian form with the Hamiltonian given by

$$H = \int (|\nabla_x \psi|^2 + U(x)|\psi|^2) dx. \tag{50}$$

In the Fourier space, Eq. (49) becomes

$$i\frac{\partial}{\partial t}\hat{\psi}_k = k^2 \hat{\psi}_k + \int \hat{U}(k-k_1)\hat{\psi}(k_1) dk_1, \tag{51}$$

with a corresponding Hamiltonian

$$H = \int \Omega(k,k_1)\hat{\psi}_k \hat{\psi}^*_{k_1} dk dk_1,$$

where $\Omega(k,k_1) = k^2 \delta^k_{k_1} + U(k_1 - k)$. Now we can apply Lemma and find that the position-dependent dispersion becomes

$$\omega_{kx} = k^2 + U(x),$$

and the corresponding Hamiltonian in terms of Gabor variables takes the canonical form (5),

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$$H_f = \int \tilde{\psi}_{kx}[\omega_{kx} - x\nabla_x\omega_{kx} + i\{\omega_{kx}, \cdot\}]\tilde{\psi}^*_{kx}dkdx. \tag{52}$$

It follows from the lemma that the Gabor variables provide a canonical description of the system (49). Indeed, since $\nabla_k\nabla_x\omega_{kx}=0$, we do not need to make a near-identity transformation (41) in the lemma. Therefore, it is instructive to obtain the same result by directly applying Gabor transform to the both sides of Eq. (49). We have

$$\Gamma[\nabla_x^2\psi] = \nabla_x^2\tilde{\psi}_{kx} + 2ik\nabla_x\tilde{\psi}_{kx} - k^2\tilde{\psi}_{kx} \approx 2ik\nabla_x\tilde{\psi}_{kx} - k^2\tilde{\psi}_{kx}. \tag{53}$$

To obtain this equation we have neglected the higher order derivative of the Gabor variable, since it is slowly varying in x. Next, we use the linear expansion of the potential $U(x_0) \approx U(x) + (x_0 - x)\nabla_x U(x)$ to find

$$\Gamma[U(x)\psi] = (U(x) - x\nabla_x U(x))\tilde{\psi}_{kx} + i\nabla_x U(x)\nabla_k\tilde{\psi}_{kx}. \tag{54}$$

Combining Eq. (53) with Eq. (54), we obtain

$$i\frac{\partial}{\partial t}\tilde{\psi}_{kx} = (k^2 + U(x) - x\nabla_x U(x))\tilde{\psi}_{kx} + i\nabla_x U(x)\nabla_k\tilde{\psi}_{kx} - 2ik\nabla_x\tilde{\psi}_{kx}.$$

Hamiltonian that corresponds to this equation is given by Eq. (52).

D. Example: An advection-type system

Let us consider an advection-type system. For simplicity of calculations let us restrict our attention to a one-dimensional case, although a general dimensional case can also be considered. An advection-type system has a Hamiltonian of the form

$$H = i\int U(x)[\psi(x)\nabla_x\psi^*(x) - \psi^*(x)\nabla_x\psi(x)]dx, \tag{55}$$

with the corresponding equation of motion

$$i\dot{\psi}(x) = -i\nabla_x(U(x)\psi(x)) - iU(x)\nabla_x\psi(x) = -i(2U(x)\nabla_x\psi(x) + \nabla_x U(x)\psi(x)). \tag{56}$$

In the Fourier space, this system is described by the Hamiltonian

$$H = \int \Omega(k,k_1)\hat{\psi}_k\hat{\psi}^*_{k_1}dkdq_{k_1},$$

with the kernel

$$\Omega(k,k_1) = (k + k_1)\hat{U}(k_1 - k). \tag{57}$$

After applying lemma to Hamiltonian (55), we obtain the following canonical form:

$$H_f = \int \check{\psi}_{kx}[\omega_{kx} - x\nabla_x\omega_{kx} + i\{\omega_{kx}, \cdot\}]\check{\psi}^*_{kx}dkdx, \tag{58}$$

where $\check{\psi}$ are the new variables.

$$\omega_{kx} = 2kU(x) \tag{59}$$

is a position-dependent frequency. Note that, in this case, we have $\nabla_k\nabla_x\omega_{kx} \neq 0$ and the near-canonical change in variables given by Eq. (41) had to be performed.

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We can also obtain the same result by directly applying the Gabor transform to Eq. (56). Using the slow dependence of $U(x)$ on x (disregarding the second derivative and higher), we obtain

$$\Gamma[U(x)\nabla_x\psi(x)] \approx (U(x) - x\nabla_x U(x) + i\nabla_x U(x)\nabla_k)(\nabla_x + ik)\tilde{\psi}_{kx}. \tag{60}$$

Similarly, we have

$$\Gamma[\nabla_x U(x)\psi(x)] \approx \nabla_x U(x)\tilde{\psi}_{kx}. \tag{61}$$

Substituting Eqs. (60) and (61) into Eq. (56), we obtain

$$i\frac{\partial}{\partial t}\tilde{\psi}_{kx} = (-i2U(x)\nabla_x + 2kU(x) + 2ix\nabla_x U(x)\nabla_x - 2xk\nabla_x U(x) + 2\nabla_x U(x)\nabla_{kx} + 2i\nabla_x U(x) + 2i\nabla_x U(x)k\nabla_k - i\nabla_x U(x))\tilde{\psi}_{kx}. \tag{62}$$

Using Eq. (59), we rewrite Eq. (62) as

$$i\frac{\partial}{\partial t}\tilde{\psi}_{kx} = \left(\omega_{kx} - x\nabla_x\omega_{kx} + \underline{\nabla_k\nabla_x\omega_{kx}\nabla_k\nabla_x} + i(\nabla_x\omega_{kx}\nabla_k - \nabla_k\omega_{kx}\nabla_x) + \underline{ix\nabla_k\nabla_x\omega_{kx}\nabla_x} + \frac{1}{2}i\nabla_k\nabla_x\omega\right)\tilde{\psi}_{kx}. \tag{63}$$

As in the lemma, we neglect the higher order terms with the two derivatives over x [underlined in Eq. (63)]. In order to obtain the canonical form of the equation of motion, we need to make a near-canonical transformation

$$\hat{\psi}_{kx} = f\check{\psi}_{kx},$$

where f satisfies

$$\nabla_k\omega_{kx}\nabla_x f - \nabla_x\omega_{kx}\nabla_k f = \tfrac{1}{2}f\nabla_k\nabla_x\omega_{kx}. \tag{64}$$

For this special case, we obtain

$$U(x)\nabla_x f = \nabla_x U(x)\left(\tfrac{1}{2}f + k\nabla_k f\right). \tag{65}$$

We have to find the solution for Eq. (65) such that $f \to 1$ when $\nabla_x U(x) \to 0$. Therefore, we need to find a solution in the form $f = 1 + g$, where g satisfies $|g(k,x)| \ll 1$. Let us try to find a solution in the form $f = f(x)$, i.e., independent of k. Then we have

$$f(x) = C\sqrt{U(x)},$$

where C is an arbitrary constant. We expand $U(x)$ around some point of reference x_0 as

$$U(x) \approx U(x_0) + (x - x_0)\nabla_x U(x_0).$$

Let us choose the constant to be $C = 1/\sqrt{U(x_0)}$, then we obtain

$$f \approx 1 + \frac{1}{2}(x - x_0)\frac{\nabla_x U(x_0)}{U(x_0)}.$$

If $\nabla_x U(x_0) \sim \varepsilon$ and $|x - x_0| \ll 1/\varepsilon$ then $f \approx 1$ and the transformation is near canonical.

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V. GENERAL CASE OF WAVES IN WEAKLY INHOMOGENEOUS MEDIA

A. Theorem

The result of the lemma can be generalized onto a much broader class of Hamiltonian given by (4). Thus, let us consider (4) with both A and B being peaked functions of $\mathbf{q}-\mathbf{q}_1$, which corresponds to waves on a weakly inhomogeneous background. Note that

$$A(\mathbf{q}_1,\mathbf{q}) = A^*(\mathbf{q},\mathbf{q}_1), \tag{66}$$

because the Hamiltonian is Hermitian. Moreover,

$$B(-\mathbf{q},-\mathbf{q}_1) = B(\mathbf{q}_1,\mathbf{q}). \tag{67}$$

Condition (67) does not really restrict our choice of the coefficient $B(\mathbf{q},\mathbf{q}_1)$. Indeed, we can consider any function B and then represent it as a sum of two components,

$$B(\mathbf{q},\mathbf{q}_1) = B'(\mathbf{q},\mathbf{q}_1) + B''(\mathbf{q},\mathbf{q}_1), \tag{68}$$

where

$$B'(\mathbf{q},\mathbf{q}_1) = \tfrac{1}{2}(B(\mathbf{q},\mathbf{q}_1) + B(-\mathbf{q}_1,-\mathbf{q})),$$

$$B''(\mathbf{q},\mathbf{q}_1) = \tfrac{1}{2}(B(\mathbf{q},\mathbf{q}_1) - B(-\mathbf{q}_1,-\mathbf{q})).$$

When we plug Eq. (68) into Hamiltonian (4), the part of the integral with B'' vanishes.

From now on, we will omit indices wherever it does not lead to confusion and denote $a \equiv \breve{a}_{\mathbf{k},\mathbf{x}}$ and $a_- \equiv \breve{a}_{-\mathbf{k},\mathbf{x}}$. Also, we introduce some convenient notations, which we will use throughout the rest of the paper. For any function $\varphi(\mathbf{k})$, we denote its even part as φ_{ev} and its odd part φ_{od},

$$\varphi_{\mathrm{ev}} = \tfrac{1}{2}(\varphi + \varphi_-),$$

$$\varphi_{\mathrm{od}} = \tfrac{1}{2}(\varphi - \varphi_-).$$

We now are ready to formulate the main theorem of this paper.

Theorem: *Consider a Hamiltonian (4) with $A(\boldsymbol{q},\boldsymbol{q}_1)$ and $B(\boldsymbol{q},\boldsymbol{q}_1)$ being peaked functions of $(\boldsymbol{q}-\boldsymbol{q}_1)$ with the same parameter ε, i.e., $A(\boldsymbol{q},\boldsymbol{q}_1)=0$ and $B(\boldsymbol{q},\boldsymbol{q}_1)=0$ when $|\boldsymbol{q}-\boldsymbol{q}_1|>\varepsilon$. Suppose that conditions (66) and (67) are satisfied. Let us introduce new notations,*

$$\mu \equiv \int A(\boldsymbol{k}-\boldsymbol{m}/2,\boldsymbol{k}+\boldsymbol{m}/2)e^{i\boldsymbol{m}\cdot\boldsymbol{x}}d\boldsymbol{m},$$

$$\lambda \equiv \int B(\boldsymbol{k}-\boldsymbol{m}/2,\boldsymbol{k}+\boldsymbol{m}/2)e^{i\boldsymbol{m}\cdot\boldsymbol{x}}d\boldsymbol{m},$$

$$\nu \equiv \Re[\lambda],$$

$$\tilde{\nu} \equiv \Im[\lambda].$$

Suppose that Hamiltonian (4) has a dominant diagonal part, i.e., $\mu_{\mathrm{ev}} > \nu$ and $\tilde{\nu}=O(\varepsilon)$. Then, there exists a new canonical change in variables from $a_{\boldsymbol{k}}$ to $c_{\boldsymbol{kx}}$, and the evolution of system (4) can be approximately described by the following equation of motion:

$$i\frac{\partial}{\partial t}c_{\boldsymbol{kx}} = \frac{\delta H_f}{\delta c^*_{\boldsymbol{kx}}},$$

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with a filtered Hamiltonian

$$H_f = \int c_{\boldsymbol{kx}}[\omega - \boldsymbol{x}\cdot\nabla_{\boldsymbol{x}}\omega + i\{\omega,\cdot\}]c^*_{\boldsymbol{kx}}d\boldsymbol{k}d\boldsymbol{x}. \tag{69}$$

The position-dependent frequency is given by the formula

$$\omega = \mu_{\mathrm{od}} + \sqrt{\mu^2_{\mathrm{ev}} - \nu^2}.$$

The transformation from the Fourier variables $a_{\boldsymbol{k}}$ to new variables $c_{\boldsymbol{kx}}$ is given in the proof.

Proof: The proof consists of three main steps. In order to diagonalize Hamiltonian (4), we (1) apply the Lemma to simplify the Hamiltonian using the peakness of the kernels, (2) perform Bogolyubov transformation to diagonalize the $O(1)$ part of the Hamiltonian, and (3) make a near-identity canonical transformation to diagonalize the $O(\varepsilon)$ part of the Hamiltonian.

Step 1: (Applying lemma) Similarly to Eq. (26), we can write a filtered Hamiltonian for Eq. (4) as

$$H^{(1)}_f = \int \breve{a}(\mu - \mathbf{x}\cdot\nabla_{\mathbf{x}}\mu + i\{\mu,\cdot\})\breve{a}^* d\mathbf{k}d\mathbf{x} + \frac{1}{2}\int [\breve{a}[\lambda - \mathbf{x}\cdot\nabla_{\mathbf{x}}\lambda + i\{\lambda,\cdot\}]\breve{a}_- d\mathbf{k}d\mathbf{x} + \text{c.c.}], \tag{70}$$

here, as usual, c.c. stands for complex conjugate. From property (67) and definition (69) it follows that $\lambda(-\mathbf{k},\mathbf{x}) = \lambda(\mathbf{k},\mathbf{x})$.

Step 2: (Bogolyubov transformation) In this step we apply the usual Bogolyubov transformation. Before doing that notice that Hamiltonian (70) consists of two parts,

$$H^{(1)}_f = H^{(1)}_{f,1} + H^{(1)}_{f,\varepsilon},$$

where

$$\begin{aligned} H^{(1)}_{f,1} &= \int \mu|\breve{a}|^2 d\mathbf{k}d\mathbf{x} + \tfrac{1}{2}\int \nu[\breve{a}\breve{a}_- + \breve{a}^*\breve{a}^*_-]d\mathbf{k}d\mathbf{x}, \\ H^{(1)}_{f,\varepsilon} &= H^{(1)}_f - H^{(1)}_{f,1}, \end{aligned} \tag{71}$$

are correspondingly $O(1)$ and $O(\varepsilon)$ parts of $H^{(1)}_f$. In Step 2, we diagonalize the $O(1)$ part using the following linear transformation:

$$\breve{a} = u_{\mathbf{kx}}b + v_{\mathbf{kx}}b^*_-. \tag{72}$$

It was shown in Ref. 10 that transformation (72) is canonical if the following conditions are satisfied:

$$|u_{\mathbf{kx}}|^2 - |v_{\mathbf{kx}}|^2 = 1,$$

$$u_{\mathbf{kx}}v_{-\mathbf{k},\mathbf{x}} = u_{-\mathbf{k},\mathbf{x}}v_{\mathbf{kx}}.$$

Let us follow Ref. 10 and choose

$$u_{\mathbf{kx}} = \cosh(\xi_{\mathbf{kx}}), \tag{73}$$

$$v_{\mathbf{kx}} = \sinh(\xi_{\mathbf{kx}}), \tag{74}$$

where $\xi_{\mathbf{kx}}$ is real and even, but otherwise arbitrary function. Then under change in variables given by Eq. (72), $H^{(1)}_{f,1}$ becomes

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$$H_{f,1}^{(1)} = \int [\mu \cosh^2(\xi) + \mu_- \sinh^2(\xi) + 2\nu \sinh(\xi)\cosh(\xi)]|b|^2 d\mathbf{k}d\mathbf{x} + \frac{1}{2}\int [(\mu + \mu_-)\sinh(\xi)\cosh(\xi) + \nu(\cosh^2(\xi) + \sinh^2(\xi))](bb_- + b^* b_-^*) d\mathbf{k}d\mathbf{x}.$$

Denote the expression in square brackets multiplying $|b|^2$ as ω,

$$\omega = \mu \cosh^2(\xi) + \mu_- \sinh^2(\xi) + 2\nu \sinh(\xi)\cosh(\xi).$$

Using trigonometric formulas for hyperbolic functions, we obtain

$$\omega = \mu_{\text{ev}} \cosh(2\xi) + \mu_{\text{od}} + \nu \sinh(2\xi). \tag{75}$$

In order to diagonalize $H_{f,1}^{(1)}$, we require that the following condition in satisfied:

$$\mu_{\text{ev}} \sinh(\xi)\cosh(\xi) + \nu(\cosh^2(\xi) + \sinh^2(\xi)) = 0. \tag{76}$$

This condition is equivalent to

$$\tanh(2\xi) = -\frac{\nu}{\mu_{\text{ev}}}. \tag{77}$$

Since $\mu_{\text{ev}} > \nu$, we can choose $\cosh(2\xi)$ to be positive and, therefore, we have

$$\cosh(2\xi) = \frac{\mu_{\text{ev}}}{\sqrt{\mu_{\text{ev}}^2 - \nu^2}}, \tag{78}$$

$$\sinh(2\xi) = -\frac{\nu}{\sqrt{\mu_{\text{ev}}^2 - \nu^2}}. \tag{79}$$

In Appendix B, we find the expression for ξ. Resolving Eq. (75) together with Eqs. (78) and (79), we obtain

$$\omega = \omega_{\text{ev}} + \omega_{\text{od}},$$

$$\omega_{\text{od}} = \mu_{\text{od}},$$

$$\omega_{\text{ev}} = \sqrt{\mu_{\text{ev}}^2 - \nu^2}.$$

Therefore, we have diagonalized $H_{f,1}^{(1)}$ to the form

$$H_{f,1}^{(1)} = \int b\omega b^* d\mathbf{k}d\mathbf{x}. \tag{80}$$

Next, we consider the $O(\varepsilon)$ part of the filtered Hamiltonian. In Appendix C, we show that Bogolyubov transformation (72) transforms $H_{f,\varepsilon}^{(1)}$ to the form

$$H_{f,\varepsilon}^{(1)} = \int b(-\mathbf{x}\cdot\nabla_{\mathbf{x}}\omega + i\{\omega,\cdot\})b^* + \left(\sigma bb_- + \frac{i\mu_{\text{ev}}^2}{2\nu} b\{\varphi, b_-\} + \text{c.c.}\right) d\mathbf{k}d\mathbf{x}, \tag{81}$$

where

$$\sigma = \frac{\mu_{\text{ev}}^2}{2\nu}\left(\mathbf{x}\cdot\nabla_{\mathbf{x}}\frac{\omega_{\text{ev}}}{\mu_{\text{ev}}}\right) + \frac{i}{2}\{\mu_{\text{od}}, \xi\} + \frac{i}{2}\tilde{\nu},$$

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$$\varphi = \sqrt{1 - \frac{\nu^2}{\mu_{\text{ev}}^2}} = \frac{\omega_{\text{ev}}}{\mu_{\text{ev}}}.$$

Combining Eqs. (80) and (81), we finally obtain Hamiltonian in the form

$$H_f^{\varepsilon} = \int b(\omega - \mathbf{x} \cdot \nabla_{\mathbf{x}}\omega + i\{\omega, \cdot\})b^* + \left(\sigma b b_- + \frac{i\mu_{\text{ev}}^2}{2\nu} b\{\varphi, b_-\} + \text{c.c.}\right) d\mathbf{k}d\mathbf{x}. \tag{82}$$

Step 3: (Near-identity transformation) In Step 2, we diagonalized $O(1)$ part but not all of the $O(\varepsilon)$ part. In order to diagonalize complete Hamiltonian, we use the near-identity transformation. This near-identity transformation changes variables from $b_{\mathbf{kx}}$ to $c_{\mathbf{kx}}$ by the following rule:

$$b_{\mathbf{kx}} = c_{\mathbf{kx}} + \alpha_{\mathbf{k}} c^*_{-\mathbf{k},\mathbf{x}} + \beta_{\mathbf{k}}\{\gamma_{\mathbf{k}}, c^*_{-\mathbf{k},\mathbf{x}}\}, \tag{83}$$

where we assume that $\beta_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ are $O(1)$ terms and $\alpha_{\mathbf{k}}$ and $(\beta_{\mathbf{k}}\{\gamma_{\mathbf{k}}, c^*_{-\mathbf{k}}\})$ are $O(\varepsilon)$ which makes our transformation indeed near identical. Note that $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$, and $\gamma_{\mathbf{k}}$ are functions of both $\mathbf{k}$ and $\mathbf{x}$. Nevertheless, for simplicity of notation, we do omit the dependence on $\mathbf{x}$, since it would only unnecessarily pollute the notations. In Appendix D, we derive the canonicity conditions for transformation (83). It turns out that transformation (83) is canonical if the following conditions are met:

$$\begin{aligned} \beta_{\mathbf{k}} &= \beta_{-\mathbf{k}}, \\ \gamma_{\mathbf{k}} &= \gamma_{-\mathbf{k}}, \\ \alpha_{\text{od}} &= \tfrac{1}{2}\{\gamma_{\mathbf{k}}, \beta_{\mathbf{k}}\}. \end{aligned} \tag{84}$$

Among the coefficients $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$, and $\gamma_{\mathbf{k}}$ that satisfy the canonicity conditions, we have to choose those that will diagonalize the $O(\varepsilon)$ part. In Appendix E, we show that such coefficients become

$$\begin{aligned} \alpha_{\mathbf{k}} &= -\frac{\sigma^*_{\mathbf{k}}}{\omega_{\text{ev}}} - \frac{\beta_{\mathbf{k}}}{2\omega_{\text{ev}}}\left\{\omega_{\text{od}}, \frac{\omega_{\text{ev}}}{\mu_{\text{ev}}}\right\}, \\ \beta_{\mathbf{k}} &= \frac{i\mu_{\text{ev}}^2}{2\nu\omega_{\text{ev}}}, \\ \gamma_{\mathbf{k}} &= \frac{\omega_{\text{ev}}}{\mu_{\text{ev}}}. \end{aligned} \tag{85}$$

Note that these conditions are in full correspondence with the canonicity conditions (84). The Hamiltonian in new variables up to $O(\varepsilon)$ order is

$$H_f = \int c[\omega - \mathbf{x} \cdot \nabla_{\mathbf{x}}\omega + i\{\omega, \cdot\}]c^* d\mathbf{k}d\mathbf{x}.$$

This completes the proof of the main result of this paper. □

B. Example: Nonlinear Schrödinger equation—waves on condensate

BEC is a state of matter that arises in dilute gases with large number of particles at very low temperatures.[21–26] BEC can be described by the nonlinear Schrödinger equation (also known as Gross–Pitaevskii equation[27]). Here, we apply the theorem to this well studied model. The evolution of the state function ψ is described by the following equation:

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$$i\frac{\partial\psi}{\partial t}+\Delta\psi-|\psi|^2\psi+\kappa(t)\psi=0,$$

with a corresponding Hamiltonian

$$H=\int\left(|\nabla\psi|^2+\frac{1}{2}|\psi|^4-\kappa(t)|\psi|^2\right)d\mathbf{x}.$$

The term $\kappa(t)\psi$ is introduced for convenience as will become clear later. Following Ref. 7, let us consider the amplitude-phase representation of the order parameter ψ,

$$\psi=Ae^{i\varphi}.$$

Now, we introduce Hamiltonian momentum

$$p=2A\varphi, \tag{86}$$

and rewrite Eq. (86) in terms of new canonical variables A and p as

$$\begin{aligned} A_t&=\frac{\delta H}{\delta p},\\ p_t&=-\frac{\delta H}{\delta A}, \end{aligned} \tag{87}$$

where

$$H=\int\left((\nabla A)^2+\frac{1}{2}A^4-\kappa(t)A^2+\frac{1}{4}\left(\nabla p-\frac{p\,\nabla A}{A}\right)^2\right)d\mathbf{x}. \tag{88}$$

Let us consider weak perturbations on background of a strong condensate,

$$A=A^{(0)}+A^{(1)},\quad p=p^{(0)}+p^{(1)},\quad |A^{(1)}|\ll|A^{(0)}|. \tag{89}$$

We now choose $\kappa(t)=(A^{(0)})^2$ which gives us $p^{(0)}\sim\varepsilon$. Substituting Eq. (89) into Eq. (88) we have

$$H=H_0+H_2+H_3,$$

where the subscripts denote the order of the term with respect to perturbation amplitudes. Since in this paper we study the linear dynamics, we only consider the quadratic part of the Hamiltonian,

$$H_2=\int\left((\nabla A^{(1)})^2+(A^{(0)})^2(A^{(1)})^2+\frac{1}{4}(\nabla p)^2+\frac{1}{2}\frac{p^{(0)}}{A^{(0)}}\nabla p\cdot\nabla A^{(1)}+\frac{1}{4}p(\nabla\ln A^{(0)})\cdot\nabla p\right)d\mathbf{x}.$$

Here, we used the fact that the spatial derivative adds one order in ε and we neglected the terms of the order of 2 and higher.

In order to apply the theorem to H_2, we first transform to Fourier space and then switch to normal variables. Let us denote $R=(A^{(0)})^2$, $S=p^{(0)}/A^{(0)}$, and $T=\nabla_x\ln A^{(0)}$. We have $R=O(1)$ and $S,T=O(\varepsilon)$. Transforming H_2 into Fourier space, we obtain

$$H_2=\int\left((\mathbf{k}_1\cdot\mathbf{k}_2\delta_2^1+R_{2-1})A_1A_2^*+\frac{1}{2}S_{2-1}\mathbf{k}_1\cdot\mathbf{k}_2p_1A_2^*+\frac{1}{4}(\mathbf{k}_1\cdot\mathbf{k}_2\delta_2^1-T_{2-1}\mathbf{k}_1\cdot(\mathbf{k}_2-\mathbf{k}_1))p_1p_2^*\right)d12,$$

where we used the following simplified notations: $1\equiv\mathbf{k}_1$, $2\equiv\mathbf{k}_2$, and subscript $2-1\equiv\mathbf{k}_2-\mathbf{k}_1$. Next, we switch to normal variables using the transformation

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$$A_{\mathbf{k}} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}} + a^*_{-\mathbf{k}}),$$
$$p_{\mathbf{k}} = -\frac{i}{\sqrt{2}}(a_{\mathbf{k}} - a^*_{-\mathbf{k}}). \tag{90}$$

In normal variables, H_2 reads

$$H_2 = \int \left(\left(\frac{5}{4}\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{\mathrm{l}}_2 + R_{2-1} + \frac{1}{8}T_{2-1}(\mathbf{k}_2 - \mathbf{k}_1)^2 \right) a_1 a^*_2 + \frac{1}{2}\left(\frac{3}{4}\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{\mathrm{l}}_2 + R_{2-1} - \frac{i}{2}S_{2-1}\mathbf{k}_1 \cdot \mathbf{k}_2 \right.$$
$$\left. + \frac{1}{4}T_{2-1}\mathbf{k}_1 \cdot (\mathbf{k}_2 - \mathbf{k}_1) \right) a_1 a_{-2} + \text{c.c.} \Big) d12.$$

The only part of the coefficient in the second parenthesis we are interested in is the one that satisfies Eq. (67),

$$\frac{3}{4}\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{\mathrm{l}}_2 + R_{2-1} - \frac{i}{2}S_{2-1}\mathbf{k}_1 \cdot \mathbf{k}_2 + \frac{1}{8}T_{2-1}(\mathbf{k}_2 - \mathbf{k}_1)^2.$$

Since $A^{(0)}$ and $p^{(0)}$ are slowly varying functions of $\mathbf{x}$, so are $R(\mathbf{x})$, $S(\mathbf{x})$, and $T(\mathbf{x})$. Therefore, their Fourier transforms are peaked around zero making the terms proportional to $T_{2-1}(\mathbf{k}_2 - \mathbf{k}_1)^2$ of the second order in ε, which can be neglected. Finally, we can write down the Hamiltonian in the form given in Eq. (4),

$$H_2 = \int \left(A(\mathbf{k}_1, \mathbf{k}_2) a_1 a^*_2 + \frac{1}{2}(B(\mathbf{k}_1, \mathbf{k}_2) a_1 a^*_2 + \text{c.c.}) \right) d12, \tag{91}$$

where

$$A(\mathbf{k}_1, \mathbf{k}_2) = \tfrac{5}{4}\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{\mathrm{l}}_2 + R_{2-1}$$
$$B(\mathbf{k}_1, \mathbf{k}_2) = \frac{3}{4}\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{\mathrm{l}}_2 + R_{2-1} - \frac{i}{2}S_{2-1}\mathbf{k}_1 \cdot \mathbf{k}_2. \tag{92}$$

In terms of window transformations, which we denote here as a, the Hamiltonian reads

$$H_f = \int a(\mu - \mathbf{x} \cdot \nabla \mu + i\{\mu, \cdot\}) a^* d\mathbf{k} d\mathbf{x} + \frac{1}{2}\int [a[\lambda - \mathbf{x} \cdot \nabla \lambda + i\{\lambda, \cdot\}] a_ d\mathbf{k} d\mathbf{x} + \text{c.c.}],$$

where

$$\mu = \int e^{i\mathbf{m}\cdot\mathbf{x}} A(\mathbf{k} - \mathbf{m}/2, \mathbf{k} + \mathbf{m}/2) d\mathbf{m} = \frac{5}{4}\mathbf{k}^2 + R(\mathbf{x}),$$
$$\lambda = \frac{1}{2}\int e^{i\mathbf{m}\cdot\mathbf{x}} B(\mathbf{k} - \mathbf{m}/2, \mathbf{k} + \mathbf{m}/2) d\mathbf{m} = \frac{3}{4}\mathbf{k}^2 + R(\mathbf{x}) - \frac{i}{2}\mathbf{k}^2 S(\mathbf{x}) + \frac{1}{2}\mathbf{k} \cdot \nabla S(\mathbf{x}). \tag{93}$$

Up to the first order in ε, we have

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$$\nu = \tfrac{3}{4}\mathbf{k}^2 + R(\mathbf{x}),$$
$$\tilde{\nu} = -\tfrac{1}{2}\mathbf{k}^2 S(\mathbf{x}). \tag{94}$$

Here, μ is an even function of $\mathbf{k}$ which means that $\mu_{ev}=\mu$ and $\mu_{od}=0$. Then, the position-dependent frequency of the small perturbations in the presence of the condensate becomes

$$\omega = \sqrt{\mu^2 - \nu^2} = |\mathbf{k}|\sqrt{R(\mathbf{x}) + \mathbf{k}^2}.$$

Bogolyubov's transformation, $a=ub+vb_-^*$, is given by the following coefficients:

$$u = \frac{\mu}{\sqrt{\mu^2 - \nu^2}} = \frac{5\mathbf{k}^2 + 4R(\mathbf{x})}{4|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}},$$

$$v = -\frac{\nu}{\sqrt{\mu^2 - \nu^2}} = -\frac{3\mathbf{k}^2 + 4R(\mathbf{x})}{4|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}}.$$

In terms of variables b, the Hamiltonian takes the following form:

$$H_f = \int b(\omega - \mathbf{x}\cdot\nabla_{\mathbf{x}}\omega + i\{\omega,\cdot\})b^* d\mathbf{k}d\mathbf{x} + \left(\int \sigma b b_- d\mathbf{k}d\mathbf{x} + \frac{i}{2}\int\left[\frac{b\mu_{ev}^2}{\nu}\{\varphi, b_-\}\right]d\mathbf{k}d\mathbf{x} + \text{c.c.}\right), \tag{95}$$

where

$$\sigma = \frac{\mu^2}{2\nu}\mathbf{x}\cdot\nabla\sqrt{1 - \frac{\nu^2}{\mu^2}} + \frac{i}{2}\tilde{\nu} = -\frac{\mathbf{k}^2}{4}\left(\frac{\mathbf{x}\cdot\nabla R(x)}{|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}} + iS(\mathbf{x})\right),$$

$$\varphi = \sqrt{1 - \frac{\nu^2}{\mu^2}} = \frac{4|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}}{5\mathbf{k}^2 + 4R(\mathbf{x})}.$$

Finally, we perform the near-identity transformation $b=c+\alpha c_-^*+\beta\{\gamma, c_-^*\}$, where

$$\alpha = \frac{\mathbf{x}\cdot\nabla R(\mathbf{x})}{4(\mathbf{k}^2 + R(\mathbf{x}))} - \frac{i|\mathbf{k}|S(\mathbf{x})}{4\sqrt{\mathbf{k}^2 + R(\mathbf{x})}},$$

$$\beta = \frac{i(5\mathbf{k}^2 + 4R(\mathbf{x}))^2}{8|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}(3\mathbf{k}^2 + 4R(\mathbf{x}))},$$

$$\gamma = \frac{4|\mathbf{k}|\sqrt{\mathbf{k}^2 + R(\mathbf{x})}}{5\mathbf{k}^2 + 4R(\mathbf{x})}.$$

The resulting Hamiltonian attains the canonical form (5).

VI. CONCLUSIONS

We have studied the dynamical behavior of the linearized spatially inhomogeneous Hamiltonian wave systems. The canonical transformation from the Fourier variables to the new spatially dependent variables is found for the general class of the quadratic Hamiltonians. In the new variables, the linearized dynamics is governed by the canonical diagonal Hamiltonian with the spatially dependent dispersion relation. The waveaction transport equation which corresponds to this Hamiltonian has form (6) which is typical to WKB formalism. It was previously obtained for some specific examples, e.g., in plasmas[28] and geophysical waves.[16,29] In this paper, we have

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given several representative examples illustrating the general results, such as the nonlinear Schrödinger equations without and with condensate and an advective-type system. Further possible areas of application of this formalism include water waves on lakes with variable depth or/and presence of variable mean flow, internal waves in media with variable stratification, plasma waves on profiles with variable density, geophysical waves in media with variable background rotation rates, etc.

The new Hamiltonian formalism that is presented in this paper should be crucial for extending the WT theory to the spatially inhomogeneous systems. In the spatially homogeneous systems, quadratic term in the Hamiltonian corresponds to the first term in Eq. (69). Effect of space inhomogeneity leads to the appearance of the derivative terms in the Hamiltonian, which correspond to the slow dynamics along the rays in the $(\mathbf{k},\mathbf{x})$-space. This effect will lead to an interesting interplay of inhomogeneity and nonlinearity in WT systems. More specifically, linear dispersion relation becomes spatially dependent. Consequently, the resonance conditions change as waves propagate through inhomogeneous environment. As a result of this, waves will remain in resonance for a *limited* amount of time, or the members of the resonant triads will change from position to position. The physical implication of this effect may be the weakened flux of energy or other conserved quantities through the wavenumber space. Other effect may be an effective broadening of the resonances, as resonances will be altered from place to place, so the wavepacket propagating through the inhomogeneous environment will be affected by averaged dispersion relation. Another potentially interesting effect is an effective three-wave interactions in a four-wave weak turbulence systems, where the role of fourth wave is played by inhomogeneity.

In order to develop a WT theory for spatially inhomogeneous systems, the kinetic equation has to be obtained for the cases with such finite-time wave resonances. This is an exciting task for the future work.

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APPENDIX A: WINDOW TRANSFORM

Let us consider the right-hand side of Eq. (33) term by term.

(1) For $\mathbf{f_0F_0}$,

$$\left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{k},\mathbf{m}) \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1$$
$$= \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{x}_1 \int F(\mathbf{k},\mathbf{m}) e^{i\mathbf{m}\cdot\mathbf{x}} d\mathbf{m} = \omega_{\mathbf{kx}} \tilde{a}_{\mathbf{kx}}. \tag{A1}$$

(2) For $\mathbf{f_1F_0}$,

$$\left(\frac{1}{\sqrt{\pi}}\right)^d \int \mathbf{m}\cdot\nabla_{\mathbf{p}} \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}+æ-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{k},æ) \tilde{a}_{\mathbf{p},\mathbf{x}_1} dp d\mathbf{m} d\mathbf{x}_1$$
$$= -\left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) i\mathbf{m}\cdot\mathbf{x} e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{k},\mathbf{m}) \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1$$
$$- \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} F(\mathbf{k},\mathbf{m}) \mathbf{m}\cdot\nabla_{\mathbf{p}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{x}_1 d\mathbf{m}$$
$$= -\mathbf{x}\cdot\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} + i\nabla_{\mathbf{x}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{k}}\tilde{a}_{\mathbf{kx}}. \tag{A2}$$

(3) For $\mathbf{f_0F_1}$,

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$$\left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} (\mathbf{p}-\mathbf{k}+\mathbf{m}/2)\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1$$

$$= -i\left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} i(\mathbf{p}-\mathbf{k}) \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{x}_1 \cdot \int e^{i\mathbf{m}\cdot\mathbf{x}} \nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) d\mathbf{m}$$

$$+ \left(\frac{1}{\sqrt{\pi}}\right)^d \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{x}_1 (-i) \int i/2 e^{i\mathbf{m}\mathbf{x}} \mathbf{m}\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) d\mathbf{m}$$

$$= -i\nabla_{\mathbf{k}}\omega_{\mathbf{kx}}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} - i/2(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}}. \tag{A3}$$

(4) For $\mathbf{f_1 F_1}$,

$$\int \mathbf{m}\cdot\nabla_{\mathbf{p}}\hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}+\mathbf{m}-\mathbf{k})\cdot\mathbf{x}} (\mathbf{p}-\mathbf{k})\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{m} d\mathbf{x}_1$$

$$= \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) i e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} \mathbf{x}\cdot(\mathbf{p}-\mathbf{k}) d\mathbf{p} d\mathbf{x}_1 \int -e^{i\mathbf{m}\cdot\mathbf{x}} \mathbf{m}\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) d\mathbf{m}$$

$$+ \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} d\mathbf{p} d\mathbf{x}_1 \int -e^{i\mathbf{m}\cdot\mathbf{x}} \mathbf{m}\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) d\mathbf{m}$$

$$+ \int \hat{f}((\mathbf{k}-\mathbf{p})/\varepsilon^*) e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} \nabla_{\mathbf{p}} \tilde{a}_{\mathbf{p},\mathbf{x}_1} \cdot (\mathbf{p}-\mathbf{k}) d\mathbf{p} d\mathbf{x}_1 \int -e^{i\mathbf{m}\cdot\mathbf{x}} \mathbf{m}\cdot\nabla_{\mathbf{k}} F(\mathbf{k},\mathbf{m}) d\mathbf{m}$$

$$= i(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{kx}\mathbf{x}\cdot\nabla_{\mathbf{x}}\tilde{a}_{\mathbf{kx}} + i(\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}\tilde{a}_{\mathbf{kx}} + (\nabla_{\mathbf{k}}\cdot\nabla_{\mathbf{x}})\omega_{\mathbf{kx}}(\nabla_{\mathbf{k}}\cdot\nabla_{x})\tilde{a}_{\mathbf{kx}}. \tag{A4}$$

APPENDIX B: CALCULATION OF ξ

In order to calculate ξ, we solve Eq. (78). Notice that $\cosh 2\xi > 1$ since $\mu_{\mathrm{ev}} > \nu$. Using the definition of the cosh function, we can write

$$e^{2\xi} + e^{-2\xi} = 2\frac{\mu_{\mathrm{ev}}}{\sqrt{\mu_{\mathrm{ev}}^2 - \nu^2}},$$

and denoting $t = e^{2\xi}$, one obtains a quadratic equation for t with two solutions,

$$t = \frac{\mu_{\mathrm{ev}} \pm |\nu|}{\sqrt{\mu_{\mathrm{ev}}^2 - \nu^2}}.$$

After we take into account Eq. (79), we obtain the following expression for ξ:

$$\xi = \frac{1}{4}\ln\frac{\mu_{\mathrm{ev}} - \nu}{\mu_{\mathrm{ev}} + \nu}. \tag{B1}$$

APPENDIX C: BOGOLYUBOV TRANSFORMATION OF THE $O(\varepsilon)$ PART

Here, we show how Bogolyubov transformation works on $H^{(1)}_{f,\varepsilon}$. We consider the terms of Eq. (71) starting with

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$$
\begin{aligned}
&-\int\left[\check{a}(\mathbf{x}\cdot\nabla_{\mathbf{x}}\mu)\check{a}^{*}+\frac{1}{2}(\mathbf{x}\cdot\nabla_{\mathbf{x}}\nu)(\check{a}\check{a}_{-}+\check{a}^{*}\check{a}_{-}^{*})\right]d\mathbf{k}d\mathbf{x}\\
&\quad=-\int(ub+vb_{-}^{*})(\mathbf{x}\cdot\nabla_{\mathbf{x}}\mu)(ub^{*}+vb_{-})d\mathbf{k}d\mathbf{x}\\
&\qquad+\left[\frac{1}{2}\int(\mathbf{x}\cdot\nabla_{\mathbf{x}}\nu)(ub+vb_{-}^{*})(ub_{-}+vb^{*})d\mathbf{k}d\mathbf{x}+\text{c.c.}\right]\\
&\quad=-\int b[(u^{2}+v^{2})(\mathbf{x}\cdot\nabla_{\mathbf{x}}\mu_{\text{ev}})+(u^{2}-v^{2})(\mathbf{x}\cdot\nabla_{\mathbf{x}}\mu_{\text{od}})+2uv(\mathbf{x}\cdot\nabla_{\mathbf{x}}\nu)]b^{*}d\mathbf{k}d\mathbf{x}\\
&\qquad-\int\left(uv(\mathbf{x}\cdot\nabla_{\mathbf{x}}\mu)+\frac{1}{2}(u^{2}+v^{2})(\mathbf{x}\cdot\nabla_{\mathbf{x}}\nu)\right)(bb_{-}+b^{*}b_{-}^{*})d\mathbf{k}d\mathbf{x}\\
&\quad=\int\left[-b(\mathbf{x}\cdot\nabla_{\mathbf{x}}\omega)b^{*}+\frac{\mu_{\text{ev}}^{2}}{2\nu}\left(\mathbf{x}\cdot\nabla_{\mathbf{x}}\sqrt{1-\frac{\nu^{2}}{\mu_{\text{ev}}^{2}}}\right)(bb_{-}+b^{*}b_{-}^{*})\right]d\mathbf{k}d\mathbf{x}. \qquad \text{(C1)}
\end{aligned}
$$

Here we used the following equalities:

$$
u^{2}+v^{2}=\cosh(2\xi),\quad 2uv=\sinh(2\xi),\quad u^{2}-v^{2}=1,
$$

$$
\sinh(2\xi)=\partial_{\nu}\omega,\quad \cosh(2\xi)=\partial_{\mu_{ev}}\omega,\quad 1=\partial_{\mu_{od}}\omega, \qquad \text{(C2)}
$$

$$
\sinh(2\xi)\nabla_{\mathbf{x}}\mu_{\text{ev}}+\cosh(2\xi)\nabla_{\mathbf{x}}\nu=-\frac{\mu_{\text{ev}}^{2}}{\nu}\nabla_{\mathbf{x}}\sqrt{1-\frac{\nu^{2}}{\mu_{\text{ev}}^{2}}}.
$$

Next, we consider the terms of Eq. (71) with the Poisson bracket,

$$
\begin{aligned}
\int i\check{a}\{\mu,\check{a}^{*}\}d\mathbf{k}d\mathbf{x}+\frac{1}{2}\int i\{\nu,\check{a}_{-}\}d\mathbf{k}d\mathbf{x}&=\frac{i}{2}\int(\check{a}\{\mu,\check{a}^{*}\}+\check{a}\{\nu,\check{a}_{-}\})d\mathbf{k}d\mathbf{x}+\text{c.c.}\\
&=\frac{i}{2}\int(ub+vb_{-}^{*})(\{\mu,ub^{*}+vb_{-}\}+\{\nu,ub_{-}+vb^{*}\})d\mathbf{k}d\mathbf{x}+\text{c.c.}
\end{aligned} \qquad \text{(C3)}
$$

Note the following.

- bb^{*} terms give zero because their coefficients are purely imaginary and we add c.c. values in the end.
- $b^{*}\nabla b$ terms can be obtained as c.c. of $b\nabla b^{*}$ terms.

Here, ∇ denotes a gradient either with respect to $\mathbf{x}$ or with respect to $\mathbf{k}$. Now, let us consider (C3) term by term.

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(1) $b\nabla b^*$ and $b^*\nabla b$:

$$\frac{i}{2}\int\left[u^2 b\{\mu,b^*\}+\underbrace{v^2 b^*_-\{\mu,b_-\}+vub^*_-\{\nu,b_-\}}_{\text{change } \mathbf{k}\to-\mathbf{k}}+uvb\{\nu,b^*\}\right]d\mathbf{x}d\mathbf{k}+\text{c.c.}$$
$$=\frac{i}{2}\int(b\{\omega,b^*\}-b^*\{\omega,b\})d\mathbf{x}d\mathbf{k}=i\int b\{\omega,b^*\}d\mathbf{x}d\mathbf{k}. \tag{C4}$$

Here, we used the fact that $\int b\{\omega,b^*\}d\mathbf{k}d\mathbf{x}=-\int b^*\{\omega,b\}d\mathbf{k}d\mathbf{x}$.

(2) bb_- and $b^*b^*_-$:

$$\frac{i}{2}\int\left[ubb_-\{\mu,v\}+vb^*_-b^*\{\mu,u\}+\underbrace{ubb_-\{\nu,u\}+vb^*_-b^*\{\nu,v\}}_{\text{give 0 because } \nu \text{ is even}}\right]d\mathbf{k}d\mathbf{x}+\text{c.c.}$$
$$=\frac{i}{2}\int[(bb_--b^*b^*_-)(u\{\mu,v\}-v\{\mu,u\})]d\mathbf{k}d\mathbf{x}$$
$$=\frac{i}{2}\int\left[(bb_--b^*b^*_-)\underbrace{\{\mu_{\text{ev}}+\mu_{\text{od}},\xi\}}_{\text{only } \mu_{\text{od}} \text{ survives}}\right]d\mathbf{k}d\mathbf{x}=\frac{i}{2}\int[(bb_--b^*b^*_-)\{\mu_{\text{od}},\xi\}]d\mathbf{k}d\mathbf{x}. \tag{C5}$$

Here, we have used Eq. (B1).

(3) $b\nabla b_-$ and $b^*\nabla b^*_-$:

$$\frac{i}{2}\int[uvb^*_-\{\mu,b^*\}+uvb\{\mu,b_-\}+u^2b\{\nu,b_-\}+v^2b^*_-\{\nu,b^*\}]d\mathbf{k}d\mathbf{x}+\text{c.c.}$$
$$=\frac{i}{2}\int[b\sinh(2\xi)\{\mu_{\text{ev}},b_-\}+b\cosh(2\xi)\{\nu,b_-\}]d\mathbf{k}d\mathbf{x}+\text{c.c.}$$
$$=\frac{i}{2}\int b\left[\frac{\mu_{\text{ev}}^2}{\nu}\left\{\sqrt{1-\frac{\nu^2}{\mu_{ev}^2}},b_-\right\}\right]d\mathbf{k}d\mathbf{x}+\text{c.c.} \tag{C6}$$

We used Eq. (C2) here.

Finally, the rest of $O(\varepsilon)$ terms are

$$\frac{i}{2}\int\tilde{\nu}aa_-d\mathbf{k}d\mathbf{x}+\text{c.c.}=\frac{i}{2}\int\tilde{\nu}bb_-d\mathbf{k}d\mathbf{x}+\text{c.c.} \tag{C7}$$

Combining Eqs. (C1)–(C7), we obtain the $O(\varepsilon)$ part of the Hamiltonian given by Eq. (82).

APPENDIX D: CANONICITY CONDITIONS FOR NEAR-IDENTITY TRANSFORMATION

Here, we obtain the canonicity conditions for the coefficients of the near-identity transformation (83). For the canonicity up to $O(\varepsilon)$ order, we use the equation of motion in the Hamiltonian form,

$$i\dot{b}_{\mathbf{k}}=\frac{\delta H}{\delta b^*_{\mathbf{k}}}. \tag{D1}$$

In this appendix for simplicity of notation, we skip writing $\mathbf{x}$ in the subscript of the dynamical variables. Using Eq. (83), we obtain

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$$i(\dot{c}_{\mathbf{k}} + \alpha_{\mathbf{k}}\dot{c}^*_{-\mathbf{k}} + \beta_{\mathbf{k}}\{\gamma_{\mathbf{k}}, \dot{c}^*_{-\mathbf{k}}\}) = \frac{\delta H}{\delta b^*_{\mathbf{k}}}. \tag{D2}$$

Since we are neglecting terms of the order higher than $O(\varepsilon)$, we can use the following approximation:

$$\dot{c}^*_{-\mathbf{k}} = i\frac{\delta H}{\delta c_{-\mathbf{k}}} \approx i\omega_{-\mathbf{k}}c^*_{-\mathbf{k}}. \tag{D3}$$

Combining Eqs. (D2) and (D3), we obtain

$$\frac{\delta H}{\delta b^*_{\mathbf{k}}} = \frac{\delta H}{\delta c^*_{\mathbf{k}}} - \alpha_{\mathbf{k}}\omega_{-\mathbf{k}}c^*_{-\mathbf{k}} - \beta_{\mathbf{k}}\{\gamma_{\mathbf{k}}, \omega_{-\mathbf{k}}c^*_{-\mathbf{k}}\}. \tag{D4}$$

Further, we have the chain rule in the form

$$\frac{\delta H}{\delta c^*_{\mathbf{k}}} = \int \left(\frac{\delta H}{\delta b^*_{\mathbf{q}}}\frac{\delta b^*_{\mathbf{q}}}{\delta c^*_{\mathbf{k}}} + \frac{\delta H}{\delta b_{-\mathbf{q}}}\frac{\delta b_{-\mathbf{q}}}{\delta c^*_{\mathbf{k}}}\right) d\mathbf{q}. \tag{D5}$$

Using Eq. (83), we find

$$\frac{\delta b^*_{\mathbf{q}}}{\delta c^*_{\mathbf{k}}} = \delta^{\mathbf{q}}_{\mathbf{k}},$$

$$\frac{\delta b_{-\mathbf{q}}}{\delta c^*_{\mathbf{k}}} = \alpha_{-\mathbf{q}}\delta^{\mathbf{q}}_{\mathbf{k}} - \beta_{-\mathbf{q}}\{\gamma_{-\mathbf{q}}, \delta^{\mathbf{q}}_{\mathbf{k}}\}_{\mathbf{q}},$$

where the subscript of the Poisson bracket indicates the differentiation with respect to $\mathbf{q}$. Therefore, Eq. (D5) becomes

$$\frac{\delta H}{\delta c^*_{\mathbf{k}}} = \frac{\delta H}{\delta b^*_{\mathbf{k}}} + \alpha_{-\mathbf{k}}\omega_{-\mathbf{k}}c^*_{-\mathbf{k}} + \{\gamma_{-\mathbf{k}}, \beta_{-\mathbf{k}}\omega_{-\mathbf{k}}c^*_{-\mathbf{k}}\}_{\mathbf{k}}. \tag{D6}$$

Combining Eqs. (D4) and (D6), we find

$$0 = -2\omega_{-\mathbf{k}}c^*_{-\mathbf{k}}\alpha_{od} - \beta_{\mathbf{k}}\{\gamma_{\mathbf{k}}, \omega_{-\mathbf{k}}c^*_{-\mathbf{k}}\} + \{\gamma_{-\mathbf{k}}, \beta_{-\mathbf{k}}\omega_{-\mathbf{k}}c^*_{-\mathbf{k}}\}. \tag{D7}$$

Finally, we obtain the canonicity conditions given in Eq. (84).

APPENDIX E: NEAR-IDENTITY TRANSFORMATION

Lets us first apply the near-identity transformation to the $O(1)$ part of the Hamiltonian that is given by Eq. (80),

$$\int \omega b^* b d\mathbf{k} d\mathbf{x} = \int \omega c^* c + (\omega\alpha^* cc_- + \omega\beta^* c\{\gamma^*, c_-\} + \text{c.c.}) d\mathbf{k} d\mathbf{x} + \text{h.o.t.}$$

To apply this transformation to the $O(\varepsilon)$ part we just need to substitute b with c in Eq. (81). The nondiagonal terms cancel in the Hamiltonian if

$$\int (\sigma + \omega\alpha^*)cc_- + \left(\omega\beta^* c\{\gamma^*, c_-\} + \frac{i\mu^2_{\text{ev}}}{2\nu}c\{\varphi, c_-\}\right) d\mathbf{k} d\mathbf{x} = 0. \tag{E1}$$

Let us choose

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$$\gamma = \gamma^* = \varphi = \frac{\omega_{ev}}{\mu_{ev}}. \tag{E2}$$

Then we can rewrite Eq. (E1) as

$$\int (\sigma + \omega\alpha^*)cc_- + \left((\omega_{ev} + \omega_{od})\beta^* + \frac{i\mu_{ev}^2}{2\nu}\right)c\{\varphi, c_-\}d\mathbf{k}d\mathbf{x} = 0.$$

Integrating by parts one can show that

$$\int \omega_{od}\beta^* c\{\varphi, c_-\}d\mathbf{k}d\mathbf{x} = \frac{1}{2}\int \{\omega_{od}\beta^*, \varphi\}cc_- d\mathbf{k}d\mathbf{x}.$$

Therefore, the diagonalizing condition becomes

$$\int \left(\sigma + \omega\alpha^* + \frac{1}{2}\{\omega_{od}\beta^*, \varphi\}\right)cc_- + \left(\omega_{ev}\beta^* + \frac{i\mu_{ev}^2}{2\nu}\right)c\{\varphi, c_-\}d\mathbf{k}d\mathbf{x} = 0$$

From Eq. (E3), the condition on β immediately follows

$$\beta = \frac{i\mu_{ev}^2}{2\nu\omega_{ev}}. \tag{E3}$$

In order to obtain the condition on α, we expand the second term in the integral

$$\omega\alpha^* = \omega_{od}\alpha_{od}^* + \omega_{ev}\alpha_{ev}^* + \omega_{od}\alpha_{ev}^* + \omega_{ev}\alpha_{od}^*. \tag{E4}$$

The integral over the last two terms vanishes because these functions are odd. Therefore, we consider only the other two terms. Next, we insert this expansion into Eq. (E3),

$$\int \left(\sigma + \omega_{od}\alpha_{od}^* + \omega_{ev}\alpha_{ev}^* + \frac{1}{2}\omega_{od}\{\beta^*, \varphi\} + \frac{1}{2}\beta^*\{\omega_{od}, \varphi\}\right)cc_- + \left(\omega_{ev}\beta^* + \frac{i\mu_{ev}^2}{2\nu}\right)\{\varphi, c_-\}cdkdx = 0.$$

Then, we obtain the following diagonalizing conditions on α:

$$\alpha_{ev} = -\frac{\sigma^*}{\omega_{ev}} - \frac{\beta}{2\omega_{ev}}\left\{\omega_{od}, \frac{\omega_{ev}}{\mu_{ev}}\right\}, \tag{E5}$$

$$\alpha_{od} = -\frac{1}{2}\left\{\beta, \frac{\omega_{ev}}{\mu_{ev}}\right\}. \tag{E6}$$

Let us prove that $\alpha_{od}=0$. Substituting Eq. (E3) into Eq. (E6), we obtain

$$\alpha_{od} = -\frac{i}{4}\left\{\frac{\mu_{ev}^2}{\nu\omega_{ev}}, \frac{\omega_{ev}}{\mu_{ev}}\right\}.$$

Expanding the Poisson bracket we find the following identity:

$$\left\{\frac{\mu_{ev}^2}{\nu\omega_{ev}}, \frac{\omega_{ev}}{\mu_{ev}}\right\} = \frac{\nu\{\mu_{ev}, \omega_{ev}\} + \mu\{\omega_{ev}, \nu\} + \omega_{ev}\{\nu, \mu_{ev}\}}{\nu^2\omega_{ev}}. \tag{E7}$$

According to the definition, $\omega_{ev}^2 = \mu_{ev}^2 - \nu^2$. Differentiation of both sides of the last equality with respect to $\mathbf{x}$ or $\mathbf{k}$ yields

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$$\nabla \omega_{\mathrm{ev}} = \frac{\mu_{\mathrm{ev}}}{\omega_{\mathrm{ev}}} \nabla \mu_{\mathrm{ev}} - \frac{\nu}{\omega_{\mathrm{ev}}} \nabla \nu. \tag{E8}$$

We use Eq. (E8) to show that

$$\begin{aligned} \{\mu_{\mathrm{ev}}, \omega_{\mathrm{ev}}\} &= -\frac{\nu}{\omega_{\mathrm{ev}}} \{\mu_{\mathrm{ev}}, \nu\}, \\ \{\omega_{\mathrm{ev}}, \nu\} &= \frac{\mu_{\mathrm{ev}}}{\omega_{\mathrm{ev}}} \{\mu_{\mathrm{ev}}, \nu\}. \end{aligned} \tag{E9}$$

Plugging Eq. (E9) into Eq. (E7) proves that $\alpha_{\mathrm{od}}=0$.

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