Nov 15

$$l m \ddot{\theta} + m g \sin\theta = \emptyset;$$

$$\ddot{\theta} + \frac{g}{l} \sin\theta = \emptyset; \qquad t = \gamma \cdot T$$

$$\ddot{\theta} + \sin\theta = \emptyset \qquad \theta = \emptyset; \ \lambda_{1,2} = \pm i;$$

$$\dot{\theta}\ddot{\theta} + \dot{\theta} \cdot \sin\theta = \emptyset \qquad \theta = \pi; \ \lambda_{1,2} = +1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$-1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\cancel{\frac{d}{dt} \frac{\dot{\theta}^2}{2} - \frac{d}{d\theta} \cos\theta = \emptyset}$$

$$\dot{\theta}\ddot{\theta} - \dot{\theta} \frac{d}{d\theta} \cos\theta = \emptyset$$

$$\frac{d}{dt} \frac{\dot{\theta}^2}{2} - \cos\theta = \emptyset \qquad E = \frac{\dot{\theta}^2}{2} - \cos\theta$$

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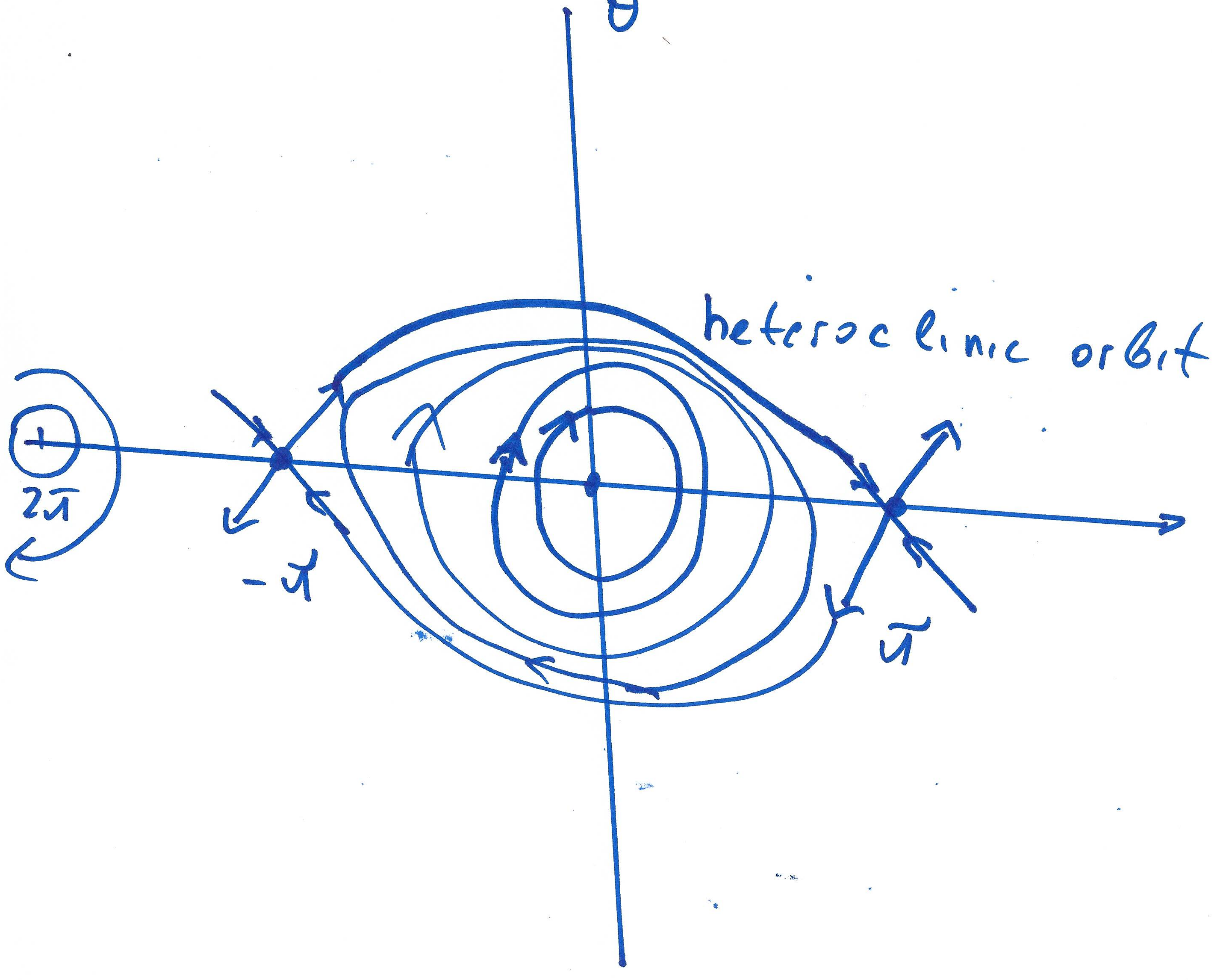

$$E = \frac{\dot{\theta}^2}{2} - \cos\theta$$

(1) $\underline{x} = (\pi, \phi)$; $E(\pi, \phi) = -\cos(\theta = \pi) = 1$

(2) $\frac{\dot{\theta}^2}{2} - \cos\theta = 1$

(3) $\frac{\dot{\theta}^2}{2} = 1 + \cos\theta$

(4) $\dot{\theta}^2 = 2(1 + \cos\theta) = 4\cos^2\theta/2$

(5) $\dot{\theta} = \sqrt{2(1 + \cos\theta)} = \frac{d\theta}{dt}$

(6) $T = \int dt = \int \frac{d\theta}{\sqrt{2(1+\cos\theta}}$

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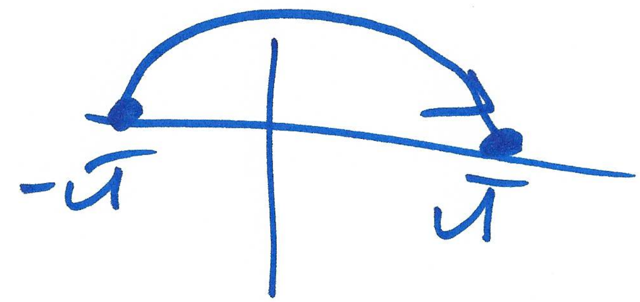
$$\dot{\theta}^2 = 4\cos^2\theta/2$$

$$\dot{\theta} = 2\cos\theta/2 = \frac{d\theta}{dt}$$



$$T = \int dt = \int_{-\pi}^{\pi} \frac{d\theta}{2\cos\theta/2}$$

$\dot{\theta} = \emptyset \Rightarrow \theta = \pi, 3\pi$

$\theta = -\pi, \pi$

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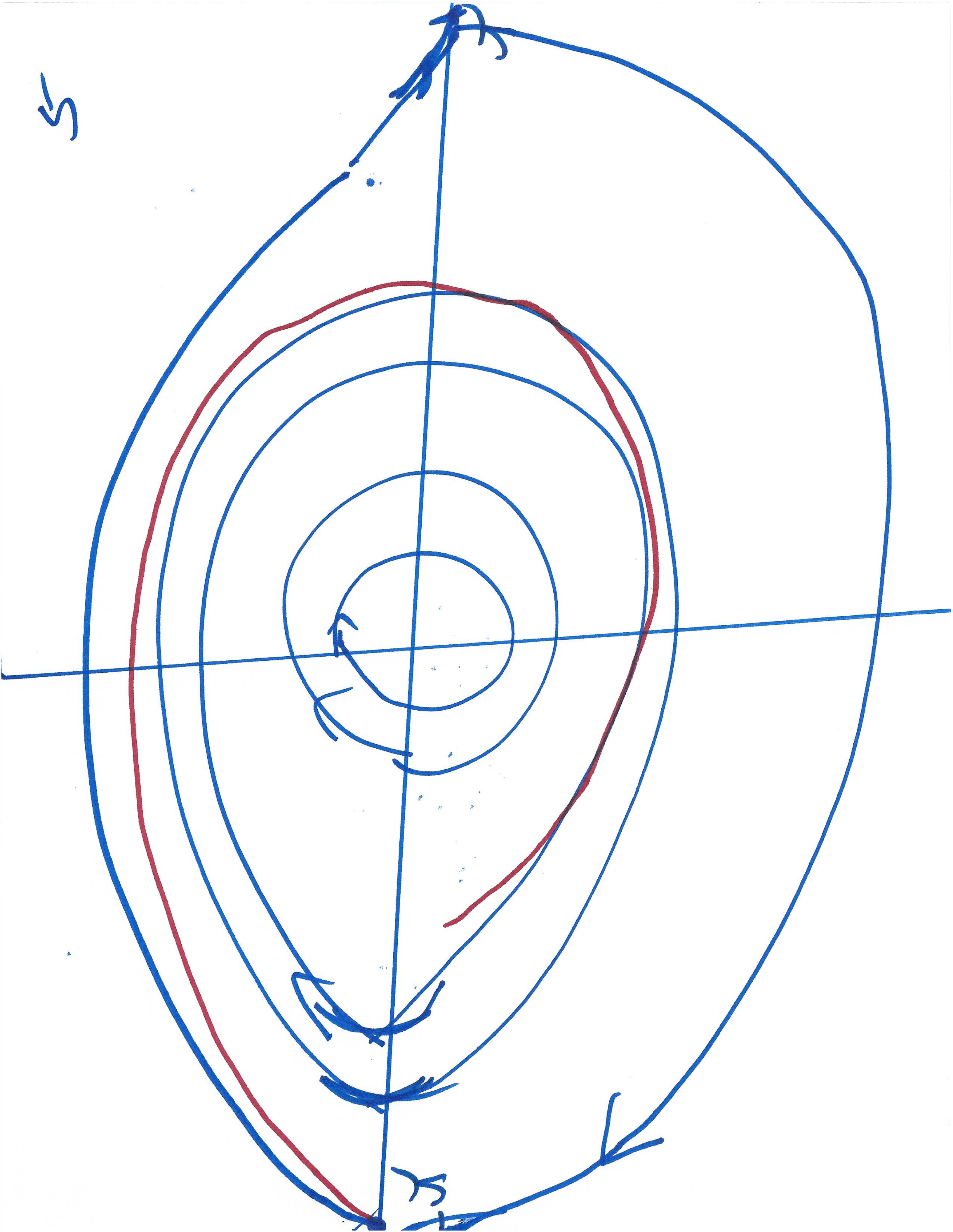


Add friction

$$\ddot{\theta} + \beta\dot{\theta} + \sin\theta = 0$$

$\hookrightarrow \beta > 0$, friction

Effects of friction on energy

$$E = \frac{\dot{\theta}^2}{2} - \cos\theta$$

$$\frac{d}{dt}E(t) = \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2} - \cos\theta\right) = \dot{\theta}\ddot{\theta} + \dot{\theta}\sin\theta =$$

$$= \dot{\theta}(\ddot{\theta} + \sin\theta) =$$

$$= \dot{\theta}(-\beta\dot{\theta}) = -\beta\dot{\theta}^2$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

nullcline: $f(x,y) = \emptyset$ or $g(x,y) = \emptyset$

conservative: $E(x,y)$ s.t.

$$\frac{d}{dt} E(x,y) = \emptyset$$

reversible systems: $t \to -t$

and other changes

$x \to -x$ $y \to -y$

$y \to -y$

$$E(x,y) = \sin(x \cdot y)$$

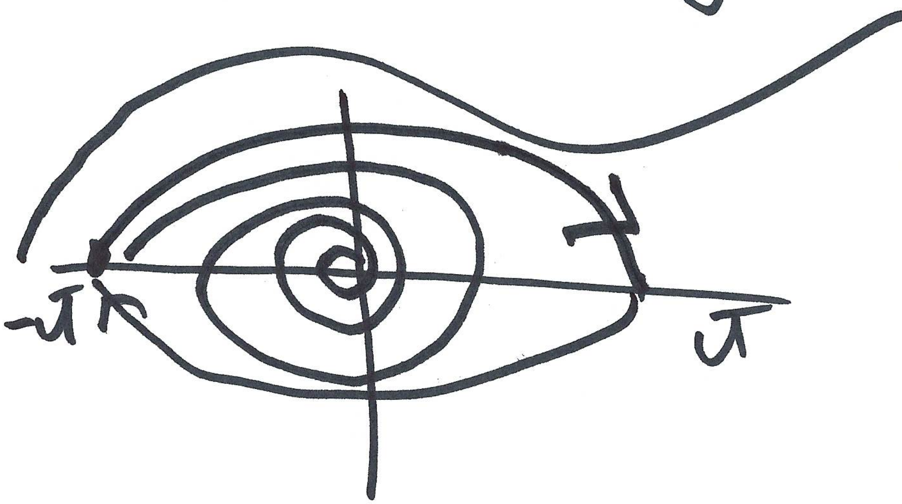
$$\emptyset = \frac{d}{dt} E(x(t), y(t))$$

$$= \frac{d}{dt} \sin(x \cdot y) = \cos(x \cdot y)(x\dot{y} + \dot{x}y) = \emptyset$$

Limit cycle.

Limit cycle is a closed and isolated trajectory.

- Closed: $\underline{x}(t) = \underline{x}(t+T)$, T is a period.
- Isolated: neighboring trajectories are not closed



pendulum with friction

$$b = \begin{cases} 0, & \text{if you are on heteroclinic trajectory} \\ 1, & \text{inside} \\ -1, & \text{outside} \end{cases}$$

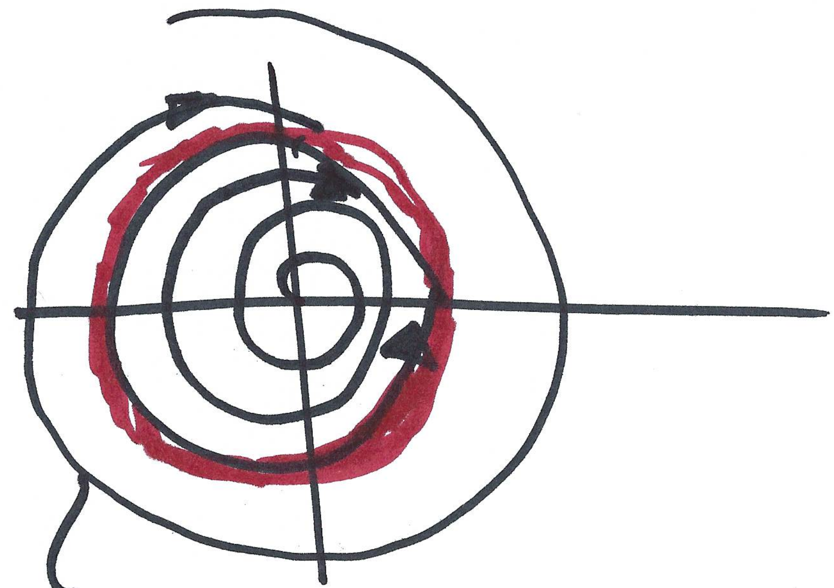
Example

$$x(t) = r(t)\cos\theta(t)$$

$$y(t) = r(t)\sin\theta(t)$$

$$-\dot{\theta}(t) = 1$$

$$\dot{r}(t) = r(t)\left(1 - r^2(t)\right)$$

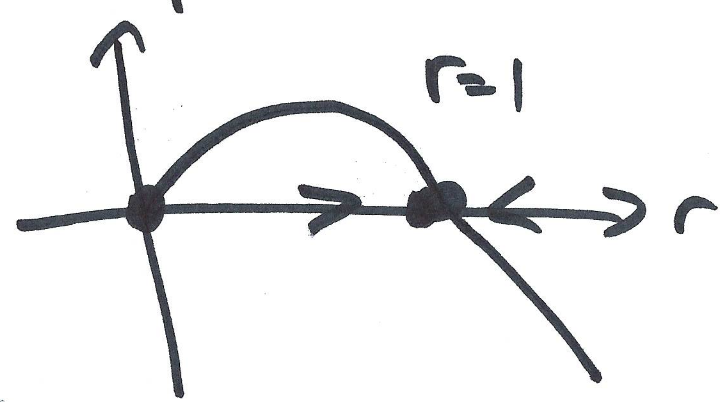


Stable limit cycle

$$r = \sqrt{x^2+y^2}; \quad \dot{r} = \frac{2(x\dot{x}+y\dot{y})}{2\sqrt{x^2+y^2}} = r(t)\left(1-r^2(t)\right)$$

$$\theta = \mathrm{ArcTan}(y/x)$$

$$\dot{\theta} = \frac{1}{1+\dots}$$



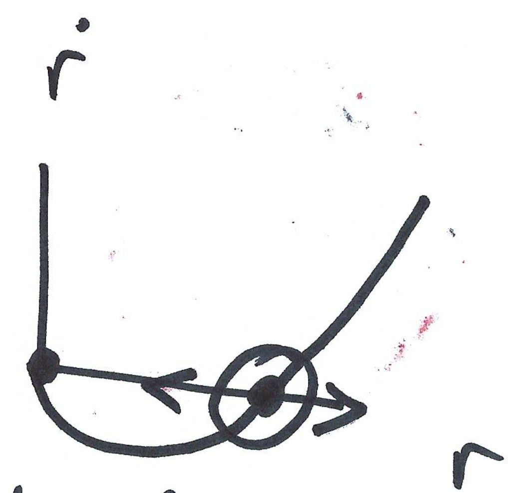
$r=1$ is stable FP of r system

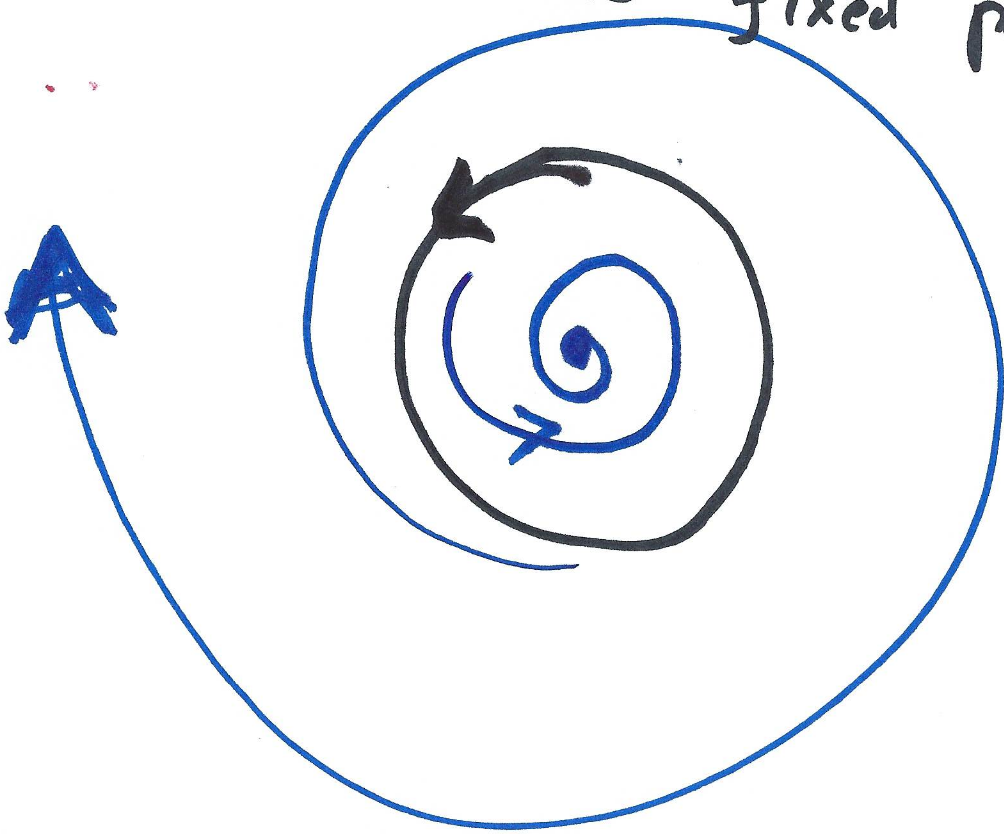
Example

$\dot{\theta} = 1$

$\dot{r}(t) = r(t)(r^2(t) - 1)$

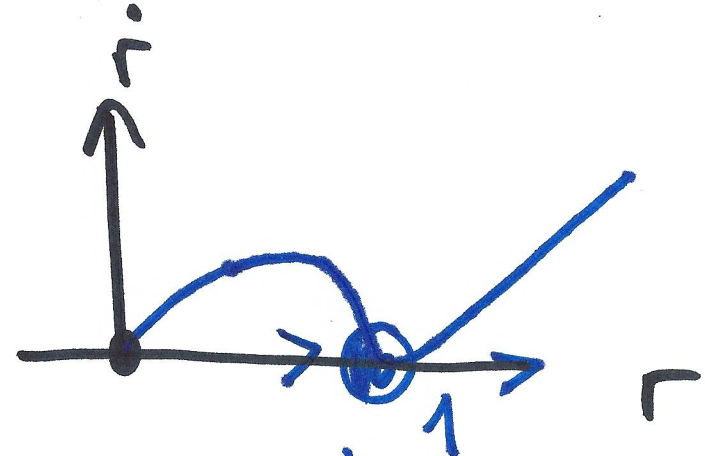
$r = 1$ is unstable fixed point of r system



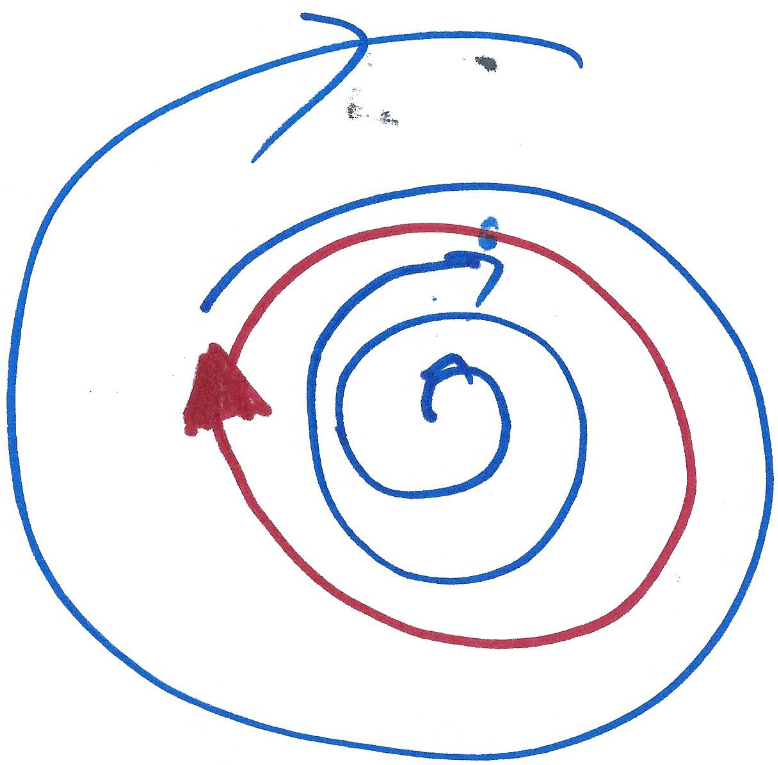


unstable limit cycle

$$\begin{cases} \dot{r} = r(1-r^2)^2 \\ -\dot{\theta} = 1 \end{cases}$$



mixed stability



Van der Paul oscillator

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = \emptyset$$

"normal" oscillator

nonlinear

$$\mu\dot{x} \leftarrow \underbrace{\mu(x^2-1)}_{\text{nonlinear friction coefficient}}\dot{x}$$

$$\mu \leftarrow \mu(x^2-1)$$

$$E(t) = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$$

$$\dot{E}(t) = \dot{x}\ddot{x} + x\dot{x} = \dot{x}(\ddot{x}+x) =$$

$$= -\mu(x^2-1)\dot{x}^2$$

IF $|x(t)| < 1$, oscillator gains energy

$|x(t)| > 1$, oscillator loses energy.

Gradient systems:

$$\left.\begin{aligned}\frac{d}{dt}\,x(t) &= -\frac{\partial}{\partial x}\,V(x(t), y(t))\\ \frac{d}{dt}\,y(t) &= -\frac{\partial}{\partial y}\,V(x(t), y(t))\end{aligned}\right\}$$

$$V(x, y) = -\sin(x \cdot y)$$

$$\left.\begin{aligned}\dot{x} &= +\frac{\partial}{\partial x}\sin(x \cdot y) = +\cos(x \cdot y) \cdot y\\ \dot{y} &= +\frac{\partial}{\partial y}\sin(x \cdot y) = +\cos(x \cdot y) \cdot x\end{aligned}\right\}$$

Reconstruct the potential

$$\dot{x} = \cos(xy)\cdot y$$
$$\dot{y} = \cos(x\cdot y)\cdot x$$

$$\dot{x} = -\frac{\partial}{\partial x} V(x,y) = \cos(x\cdot y)\cdot y$$

arbitrary function

$$V(x,y) = -\int \cos(x\cdot y)\cdot y \, dx + F(y)$$

$$= -y\int \cos(xy)\,dx = -\frac{y}{y}\cdot \sin(x\cdot y) =$$

$$= -\sin(x\cdot y) + F(y);$$

$$\dot{y} = \cos(x\cdot y)\cdot x$$

$$= -\frac{\partial}{\partial y} V(x,y)$$

$$V(x,y) = -\int \cos(x\cdot y)\cdot x \, dy$$

$$= -\sin(x\cdot y) + G(x)$$

$$G(x) = F(y) = 0$$

$$V(x,y) = -\sin(xy)$$

Theorem:

Gradient system can not have a limit cycle.

Consider a gradient system: $\dot{x} = -\frac{\partial V}{\partial x}$, $\dot{y} = -\frac{\partial V}{\partial y}$

and calculate

$$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V(x,y)}{\partial x} \cdot \dot{x} + \frac{\partial V(x,y)}{\partial y} \cdot \dot{y} =$$

$$= -\left[\left(\frac{\partial V(x,y)}{\partial x} \right)^2 + \left(\frac{\partial V(x,y)}{\partial y} \right)^2 \right] \leq \emptyset$$

Let $\underline{x}(t)$ be a closed orbit $\underline{x}(t+T) = \underline{x}(t)$

$$\forall \quad \Delta V = V(x(t+T), y(t+T)) - V(x(t), y(t)) = \emptyset$$

$$\Delta V = \int_{t}^{t+T} \dot{V}(x(\tau), y(\tau))\, d\tau$$

$$= \int_{t}^{t+T} d\tau \cdot \frac{dV}{dt} \leq \emptyset$$

Show that there are no closed or bits $\dot{x} = \sin y$, $\dot{y} = x \cos y$

This is a gradient system:

$$\dot{x} = \sin y = -\frac{\partial V}{\partial x} \Rightarrow V(x,y) = -\int \sin y \, dx = -x \sin y + F(y)$$

$$\dot{y} = \cos y \cdot x = -\frac{\partial V}{\partial y} \Rightarrow V(x,y) = -\int (\cos y)\, x \, dy = $$

$$= -\sin y \cdot x + G(x) \qquad F = G = 0$$

$$V(x,y) = -x \sin y$$