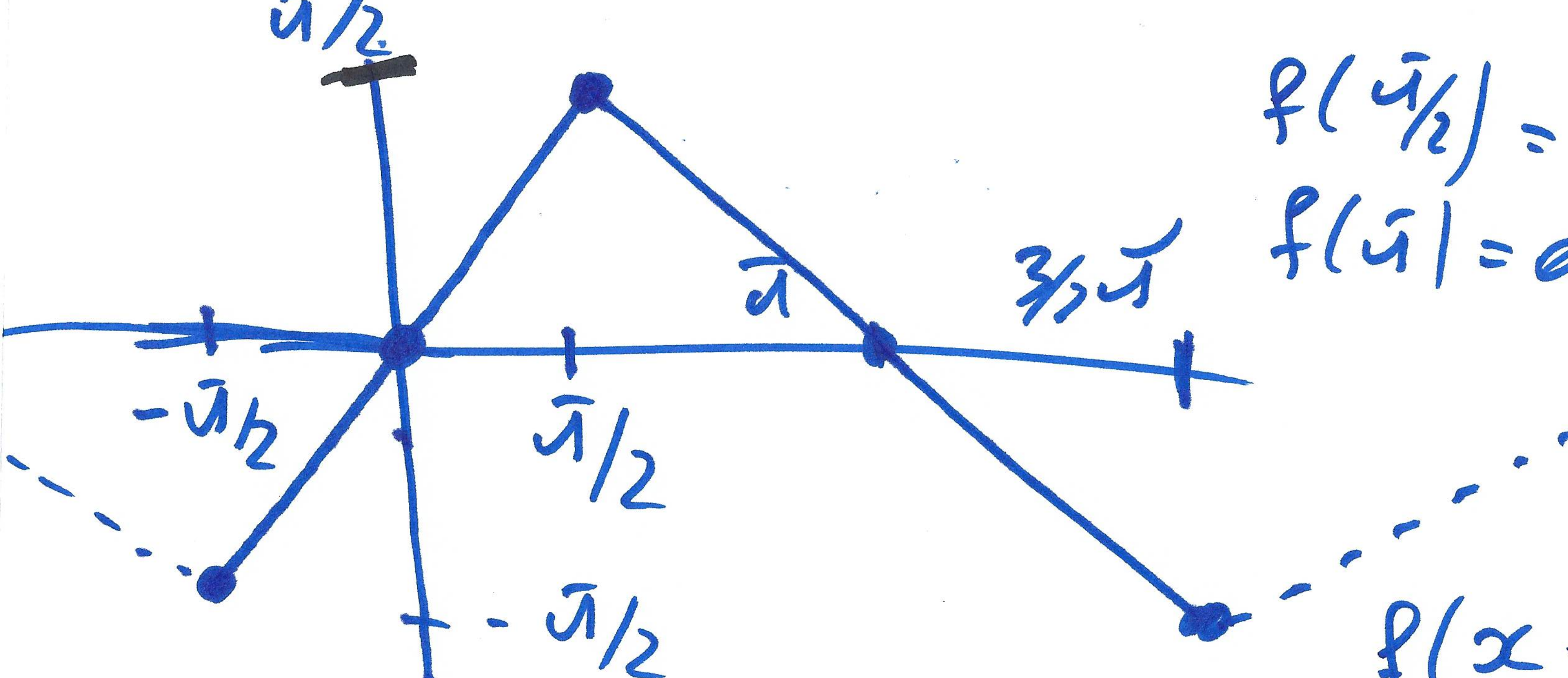
Midterm Review

$\dot{\Psi} = \Omega$ Example $\dot{z}$

$\dot{\theta} = \omega + A\, f(\Psi - \theta)$

$$f(\Psi) = \begin{cases} \Psi, & -\frac{\pi}{2} < \Psi < \frac{\pi}{2} \\ \pi - \Psi, & \frac{\pi}{2} < \Psi < \frac{3}{2}\pi \end{cases}$$

Plot $f(\Psi)$



$f(\pi/2) = \pi/2$

$f(\pi) = 0$

$f(x + 2\pi) = f(x)$

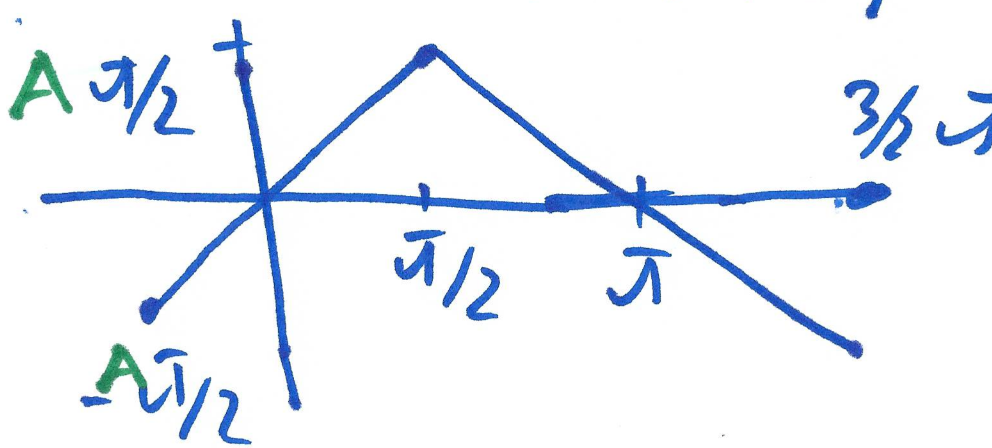
$$\dot{\psi} - \dot{\theta} = \Omega - \omega - A f(\psi - \theta)$$

$$\cancel{\dot{\theta} - \Omega}$$

$$\varphi = \psi - \theta, \quad \dot{\varphi} = \Omega - \omega - A f(\varphi);$$

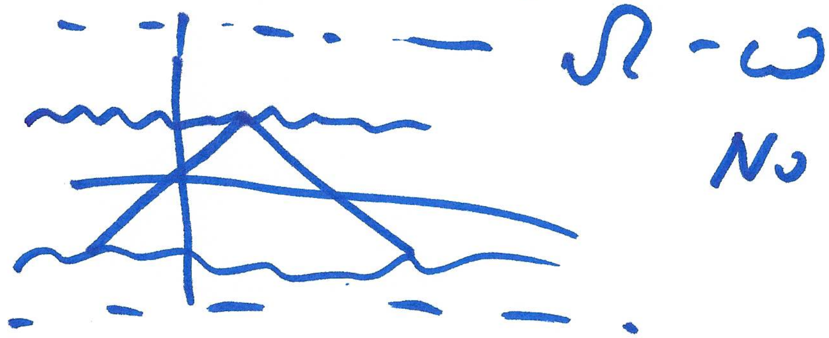
Find fixed point

$$\Omega - \omega = A f(\varphi)$$



$$-\frac{A\pi}{2} < A f(\varphi) < A\pi/2$$

- $\Omega - \omega > A J/2$



No fixed points

$\Omega - \omega < -AJ/2$ - No fixed points

$$-\frac{AJ}{2} < \Omega - \omega < AJ/2$$

$$\boxed{\omega - \frac{AJ}{2} < \Omega < \omega + AJ/2}$$

$$\dot{\varphi}(t) = \Omega - \omega - A f(\varphi)$$

Assume $\omega - \frac{\pi A}{2} < \Omega < \omega + \frac{\pi A}{2}$

Find φ such that

$$\frac{\Omega - \omega}{A} = f(\varphi)$$

$$x = f(\varphi)$$

$$f_{\text{inverse}}(x) = \varphi$$

$$\varphi^* = f_{\text{inverse}}\left(\frac{\Omega - \omega}{A}\right) = \cancel{f^{-1}\left(\frac{\Omega - \omega}{A}\right)}$$

- Assume $|\varphi^*| < \frac{\pi}{2}$

$$-\frac{\pi}{2} < \varphi^* < \pi/2 \quad f(\varphi) = \varphi;$$

$$\Omega - \omega = A f(\varphi^*) = A\varphi^*$$

$$\boxed{\varphi^* = \frac{\Omega - \omega}{A}} \quad \boxed{-\frac{\pi}{2} < \varphi^* < \frac{\pi}{2}}$$

Assume $\frac{\pi}{2} < \varphi^* < \frac{3}{2}\pi$

$$\Omega - \omega = A f(\varphi^*) = A(\pi - \varphi^*) = \pi A - A\varphi$$

$$A\varphi^* = \pi A + \Omega - \omega$$

$$\boxed{\varphi^* = \pi + \frac{\Omega - \omega}{A}}$$

$$\dot{\theta} = f(\theta);$$

$$f(\theta + 2\pi) = f(\theta)$$

uniform: $f(\theta) = \Omega$; $\dot{\theta} = \Omega \Rightarrow \theta = \Omega t$

$$\theta(t) = \Omega t \mod 2\pi$$

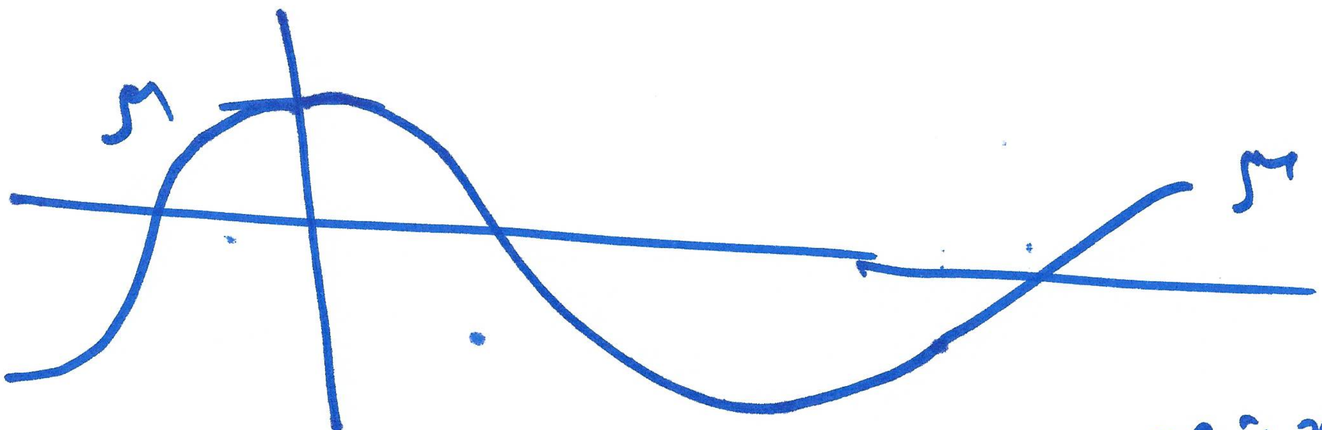
$$T = \frac{2\pi}{\Omega}$$

Nonuniform oscillator

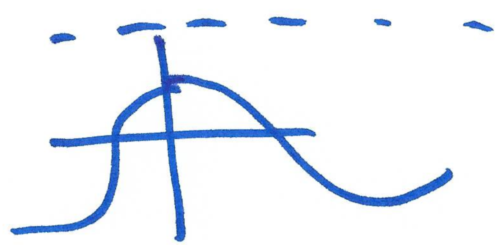
$\dot{\theta} = \omega$ [redacted] $\mu \cos\theta$

$$-1 \le \cos\theta \le 1$$

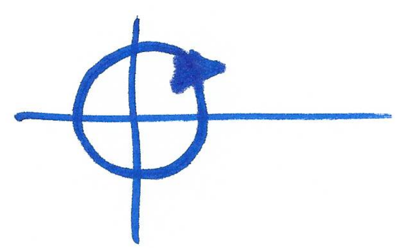
$$-\mu \le \mu\cos\theta \le \mu$$



$\omega > \mu$



periodic motion



$-\mu \leq \omega \leq \mu$

Fixed point(s) exist(s)

$\mu = \omega$ - one fixed point

$-\mu < \omega < \mu$

- 2 fixed points

$-\mu = \omega$ - one fixed point.

$\theta^* = \pm \text{Arc}\cos(\omega/\mu)$

$\dot{\theta} = \omega + \mu$

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

when $b\dot{\theta} + mgL\sin\theta = \Gamma$ is a good approximation?

$t = T\cdot\tau$; $[T]$ = time (seconds)

$[\tau]$ = dimensionless

$$\frac{mL^2}{T^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{T}\frac{d}{d\tau}\theta(\tau) + mgL\sin\theta = \Gamma$$

$$\frac{L}{T^2 g}\frac{d^2\theta}{d\tau^2} + \frac{b}{T\,mgL}\frac{d}{d\tau}\theta(\tau) + \sin\theta = \frac{\Gamma}{mgL}\quad \frac{1}{mgL}$$

choose $T = \frac{b}{mgL}$

12

$$\frac{L\, m^2 g^2 L^2}{g\, b^2} \frac{d^2\theta}{dT^2} + \frac{d}{dT}\theta(T) + \sin\theta = \frac{\Gamma}{mgL}$$

$$\varepsilon = \frac{L^3 m^2 g}{b^2}; \quad [\varepsilon] = 1$$

$$\varepsilon \frac{d^2}{dT^2}\theta(T) + \frac{d}{dT}\theta(T) + \sin\theta = \gamma,$$

$$\omega(T) = \frac{d\theta(T)}{dT} \qquad \gamma = \frac{\Gamma}{mgL}$$

$\cancel{\frac{d\omega}{dT} + \frac{1}{\varepsilon}\sin}$

$$\begin{cases} \varepsilon \frac{d}{dT}\omega + \omega(T) + \sin\theta = \gamma \\ \frac{d\theta(T)}{dT} = \omega(T) \end{cases}$$

$$\begin{cases} \dfrac{d}{d\tau}\omega(\tau) = \dfrac{1}{\varepsilon}(\gamma - \sin\theta - \omega) \\[2ex] \dfrac{d\theta}{d\tau} = \omega \end{cases}$$

$$\dot{\theta} = \frac{\sin\theta}{\mu + \cos\theta}$$

Fixed point $\dot{\theta} = \emptyset$: $\theta = \emptyset$, $\theta = \pi$;

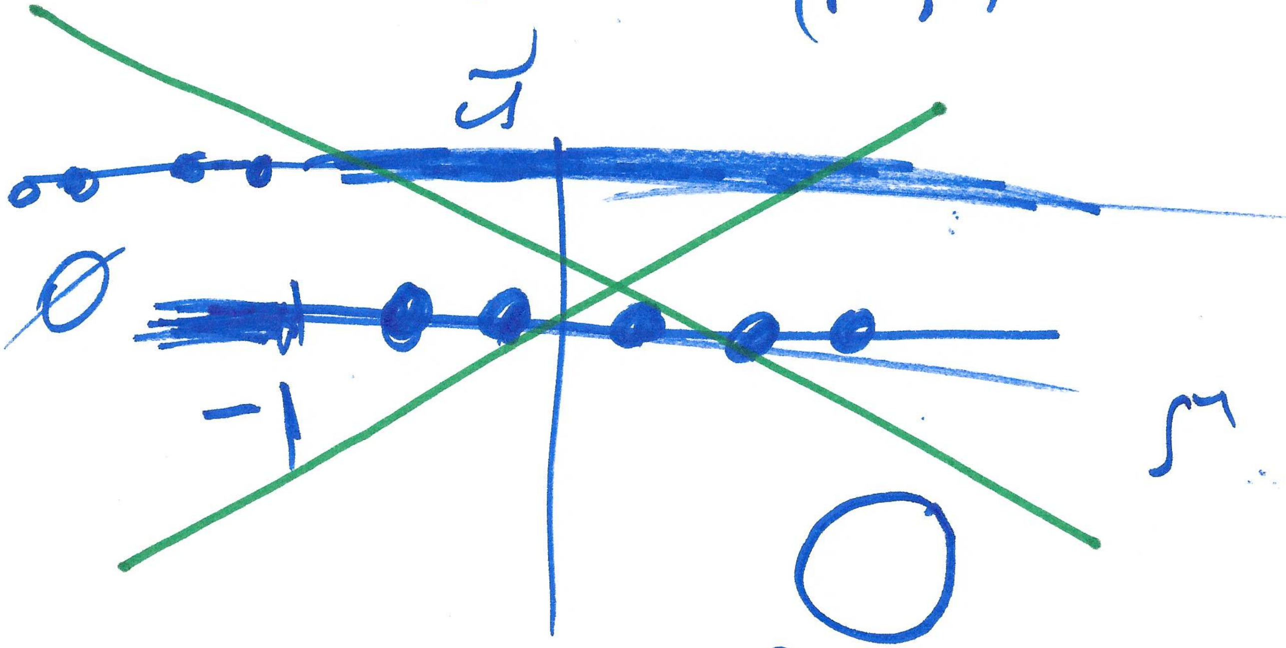
$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\mu + \cos\theta}\right) = \frac{\cos\theta}{\mu + \cos\theta} + \frac{\sin^2\theta}{(\mu + \cos\theta)^2}$$

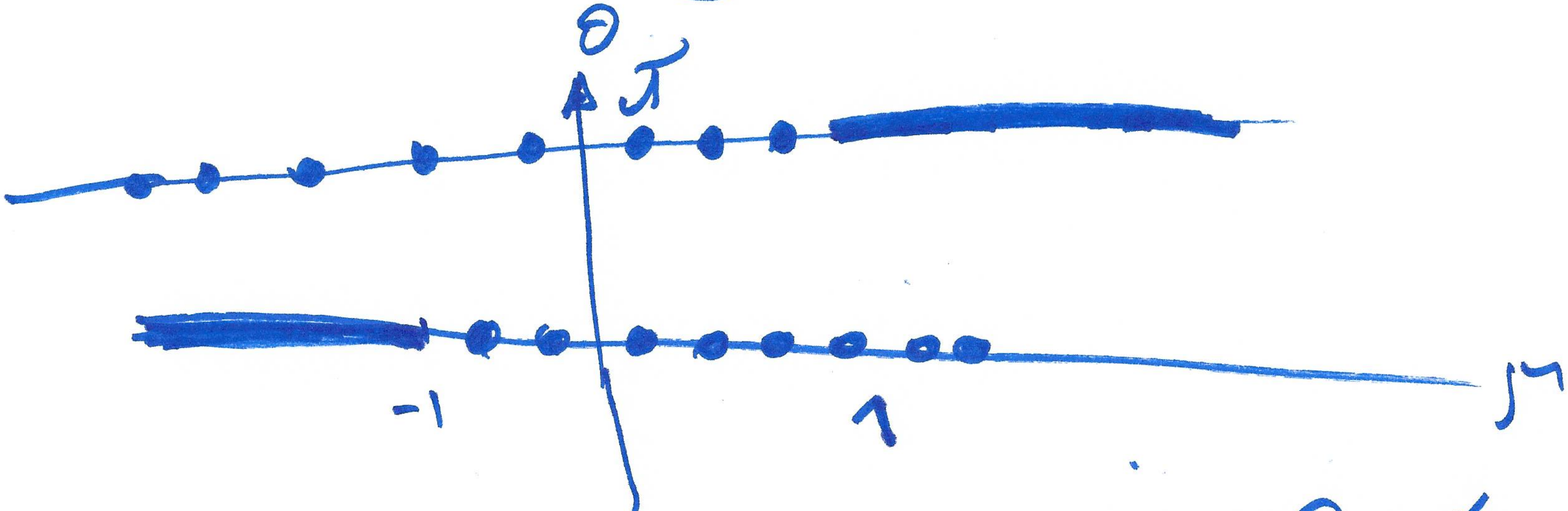
$$= \frac{(\mu + \cos\theta)\cos\theta + \sin^2\theta}{(\mu + \cos\theta)^2} =$$

$$= \frac{\mu\cos\theta + 1}{(\mu + \cos\theta)^2}$$

$$\left.\frac{d}{d\theta}(f(\theta))\right|_{\theta = \emptyset} = \frac{1+\mu}{(1+\mu)^2} = \frac{1}{1+\mu}\ ;$$ $\mu > -1$, unstable; $\mu < -1$ stable

$$\frac{d}{d\theta}\left(f(\theta)\right)\Big|_{\theta=\pi} = \frac{1-\mu}{(1-\mu)^2} = \frac{1}{1-\mu}\begin{cases} \mu > 1, & \text{stable} \\ \mu < +1 & \text{unstable}\end{cases}$$

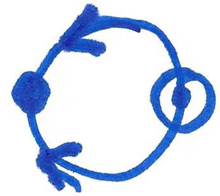




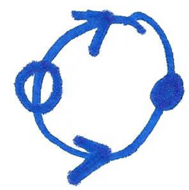
$\mu + \cos\theta = \emptyset,$

$\boxed{\theta = \mathrm{ArcCos}\,-\mu}$

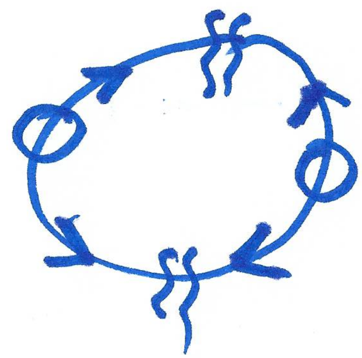
$\mu > 1$



$\mu < -1$

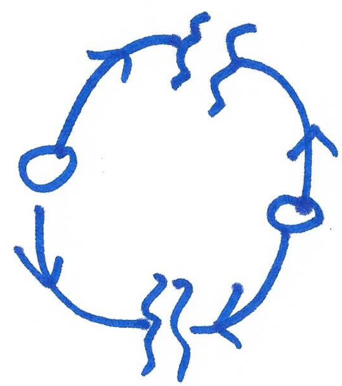


$-1 < \mu < 1$



Singularity

$\theta_{sin} = \pm \mathrm{ArcCos}\, \mu$



Period: $\dot{\theta} = f(\theta)$; $f(\theta) \neq 0$

for $0 \leq \theta \leq 2\pi$

$$\frac{d\,\theta(t)}{dt} = f(\theta)$$

$$T = \int dt = \int_0^{2\pi} \frac{d\theta(t)}{f(\theta)}$$

$$\frac{d}{dt} x(t) = a\,x(t) + b\,y(t)$$

$$\frac{d}{dt} y(t) = c\,x(t) + d\,y(t)$$

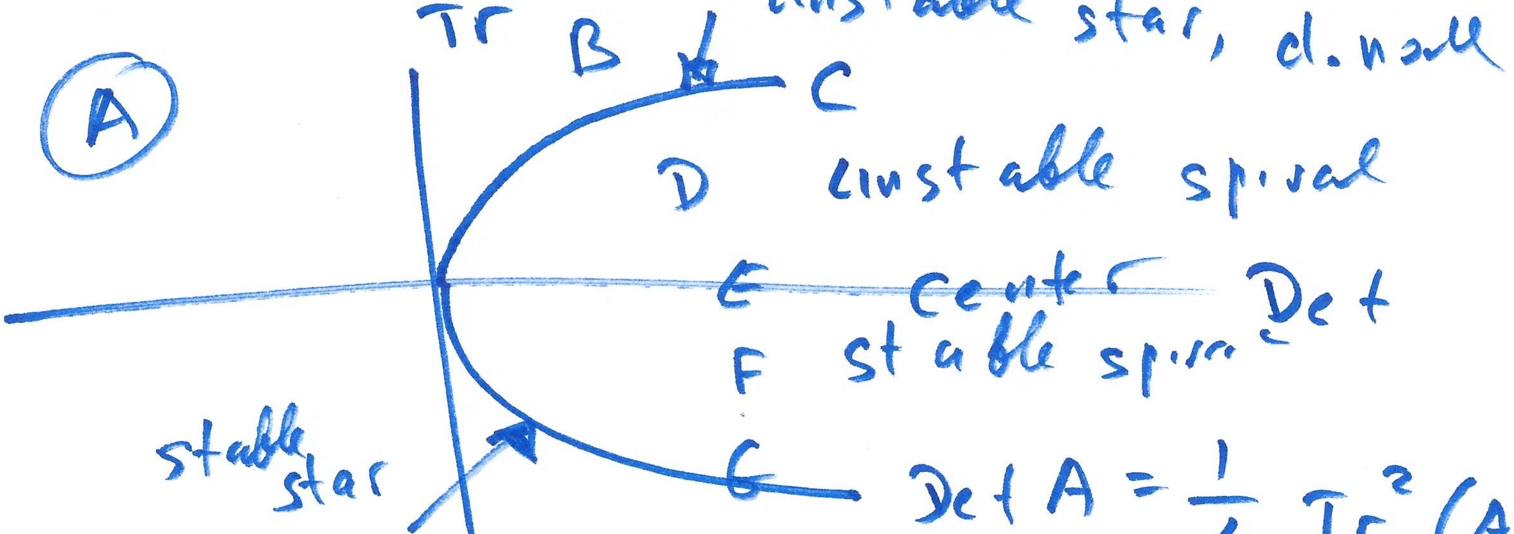
$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Eigenvalues $(\lambda - a)(\lambda - d) - bc = 0$

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \mathrm{Det}\,A = 0$$

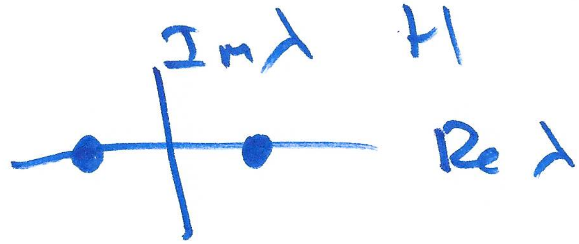
$$\lambda_{12} = \frac{\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}^2(A) - 4\,\mathrm{Det}\,A}}{2}$$

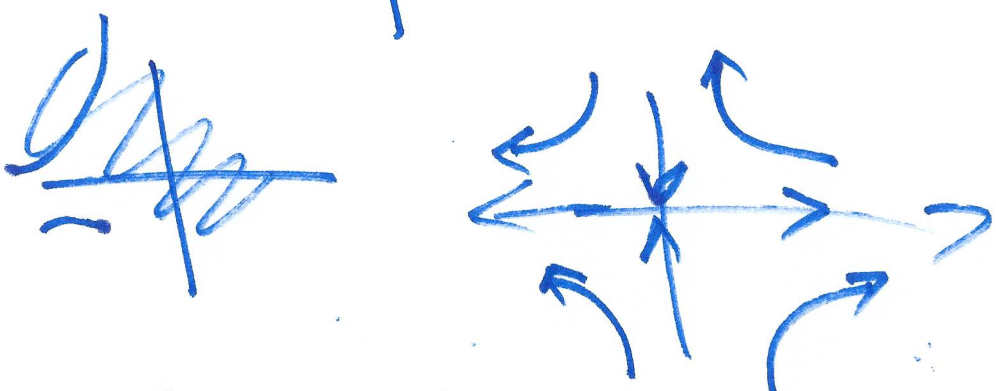
(A)



$$\mathrm{Det}\, A = \frac{1}{4} \mathrm{Tr}^2(A)$$

(A) $\mathrm{Det}(A) < 0$:

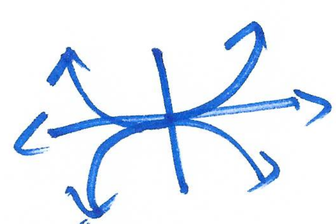




Saddle

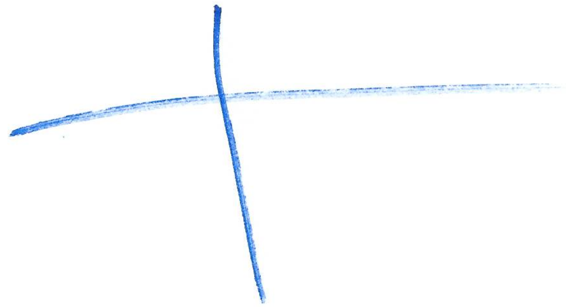
• (B) $\mathrm{Det} > 0, \mathrm{Tr} > 0, \mathrm{Tr}^2 > 4\,\mathrm{Det}$

2 positive roots



unstable node

• (H) $\mathrm{Tr} < 0, \mathrm{Det} > 0, \mathrm{Tr}^2 > 4\,\mathrm{Det}$



$\lambda_1 = \lambda_2 > 0$, 2 distinct eigenvectors