1

$$\left.\begin{aligned} \frac{d}{dt} x(t) &= f(x(t), y(t)) \\ \frac{d}{dt} y(t) &= g(x(t), y(t)) \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{aligned}\right\}$$

nucline

① $\dot{x} = 0$ → vertical

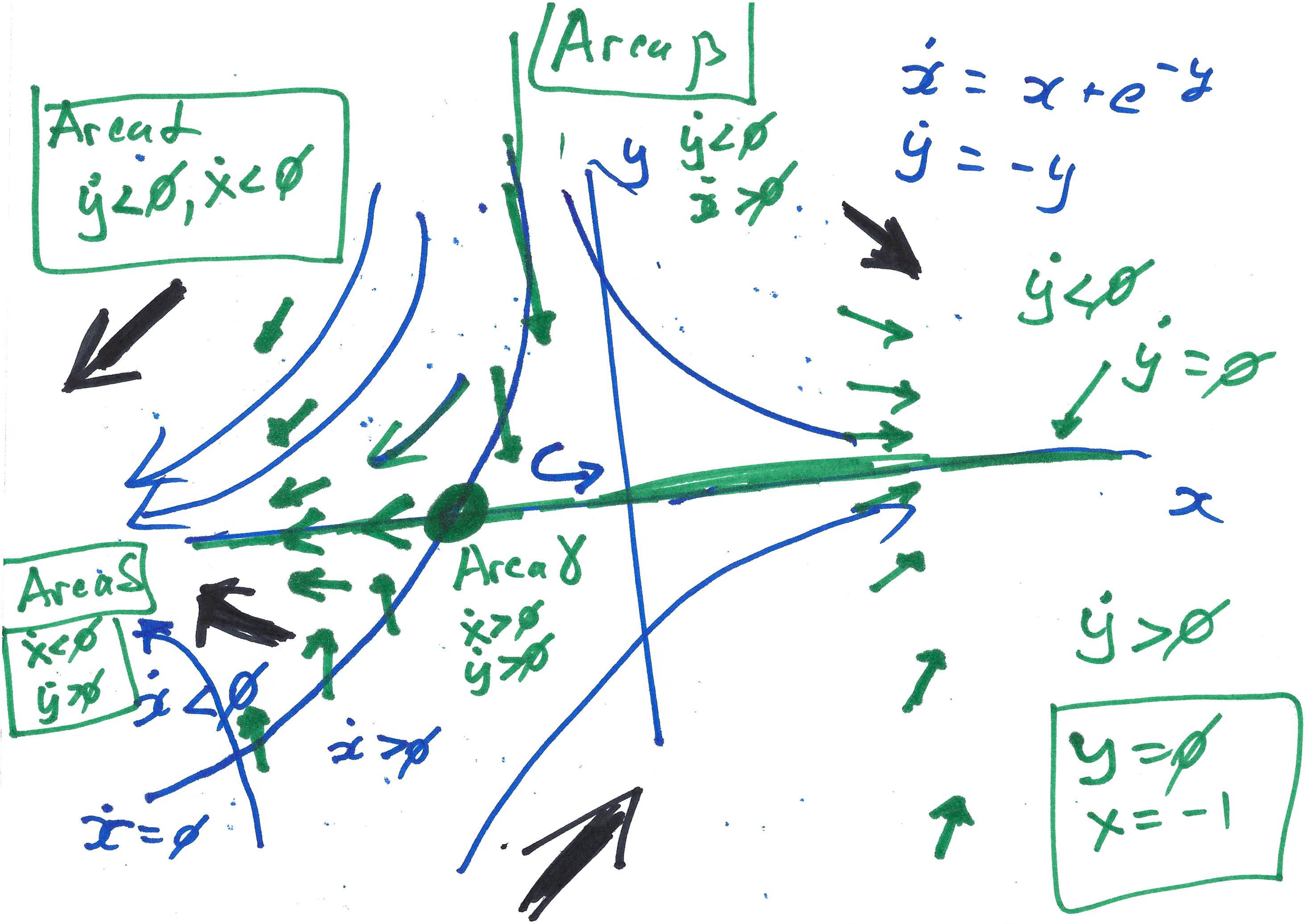
② $\dot{y} = 0$ → purely horizonta

$\dot{x} = 0 \Rightarrow x = -e^{-y}$

$\dot{y} = 0 \Rightarrow y = 0$

$\dot{x} = x + e^{-y}$

$\dot{y} = -y$



$y = 0$

$x = -1$

$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$
$$x^* = -1,\ y = \emptyset;$$
$$x(t) = -1 + \nu(t)$$
$$y(t) = 0 + \mu(t)$$

$$\frac{\partial f}{\partial x} = 1;\quad \frac{\partial f}{\partial y} = -e^{-y};\quad \frac{\partial f(-1,\emptyset)}{\partial y} = -$$
$$\frac{\partial g}{\partial x} = \emptyset;\quad \frac{\partial g}{\partial y} = -1$$

$$\frac{d}{dt}\begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix} = \left.\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}\right|_{(-1,\emptyset)} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & -1 \\ \emptyset & -1 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix};\quad \begin{matrix} \text{Det} = -1 \\ \text{Tr} = \emptyset \end{matrix}$$

$$\lambda_{1,2} = \frac{Tr \pm \sqrt{Tr^2 - 4\,Det}}{2}$$

$$= \frac{0 \pm \sqrt{4}}{2} = \pm 1$$

$\lambda_1 = 1$

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$a - b = a, \quad b = 0, a = 1$

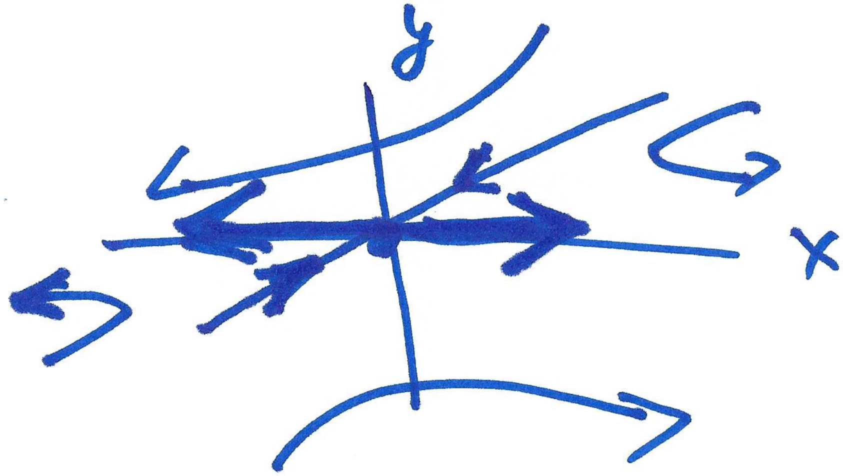
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\boxed{\lambda_2 = -1}$

$c - d = -c$

$d = 2c$

$v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$


y
x

Existence and uniqueness

Consider
$$\left.\begin{aligned}\frac{d}{dt}\underline{x}(t) &= \underline{f}(\underline{x}(t)) \\ (*)\quad \underline{x}(t=t_0) &= \underline{x}_0\end{aligned}\right\}(*)$$
initial value problem

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}; \quad \underline{f}(\underline{x}) = \begin{pmatrix} f_1(x_1 \ldots x_n) \\ f_2(x_1 \ldots x_n) \\ \vdots \\ f_n(x_1 \ldots x_n) \end{pmatrix}$$

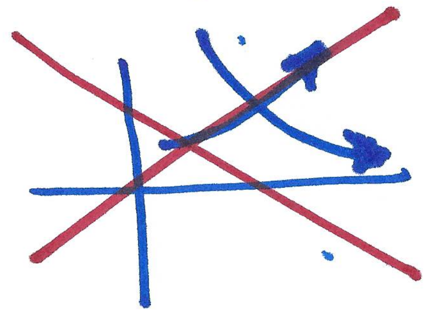
IF $\underline{f}(\underline{x})$ is continious along with its first derivatives "around" x_0

↳ on some open interval

Then solution to IVP (*) EXISTS and is unique for some region around t_0.

Why we care?

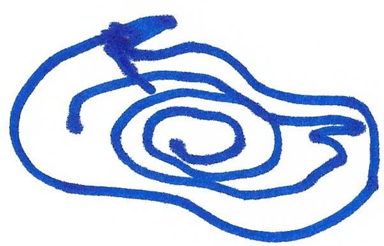
① trajectories never cross



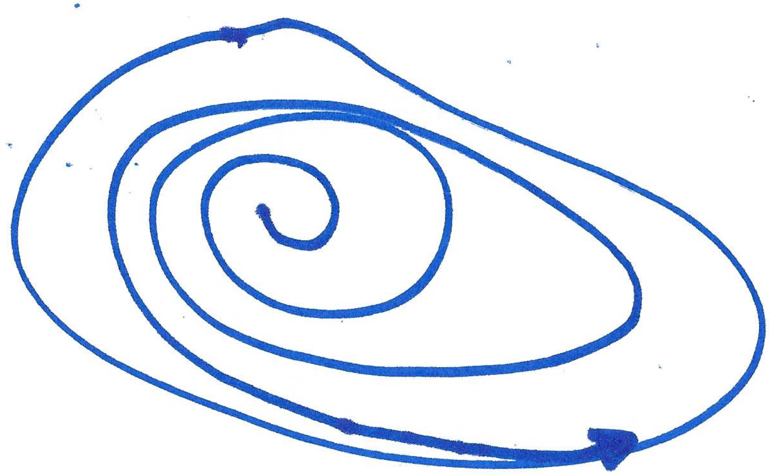
If they were to cross, the solution would not be unique

②  trajectories are smooth curves

③ Suppose that closed (period) trajectories exist. $\underline{X}(t) = \underline{X}(t+T)$, $T \neq 0$.



IF start inside, always stay inside

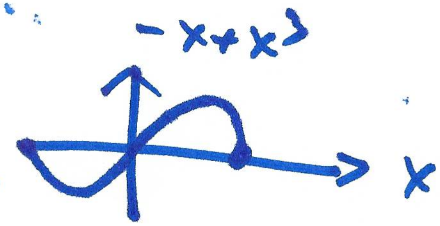


Example.

$$\begin{cases} \dot{x} = -x + x^3 \\ \dot{y} = -2y \end{cases} \qquad J(x,y) = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

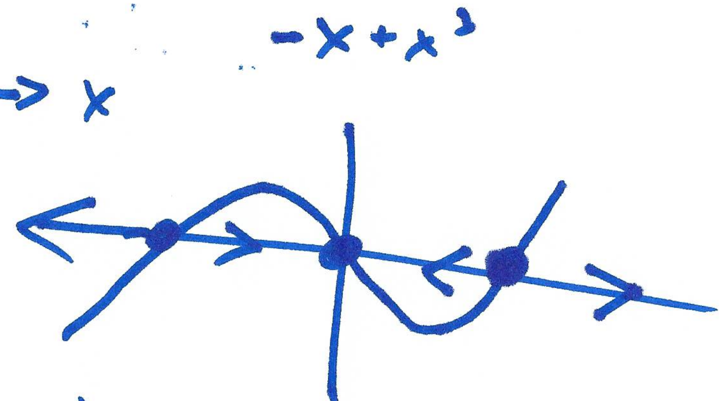
Find fixed points, linearize around them, study stability, plot phase portrait.

- $\dot{x} = -x + x^3$



$x = 0, \; x = \pm 1$

$x = 0$: stable; $x = \pm 1$: unstable



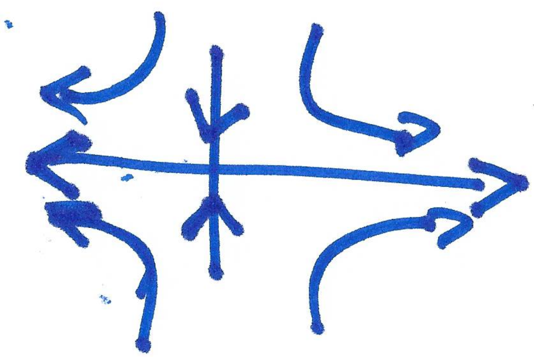
- $\dot{y} = 0 \Rightarrow y = 0$

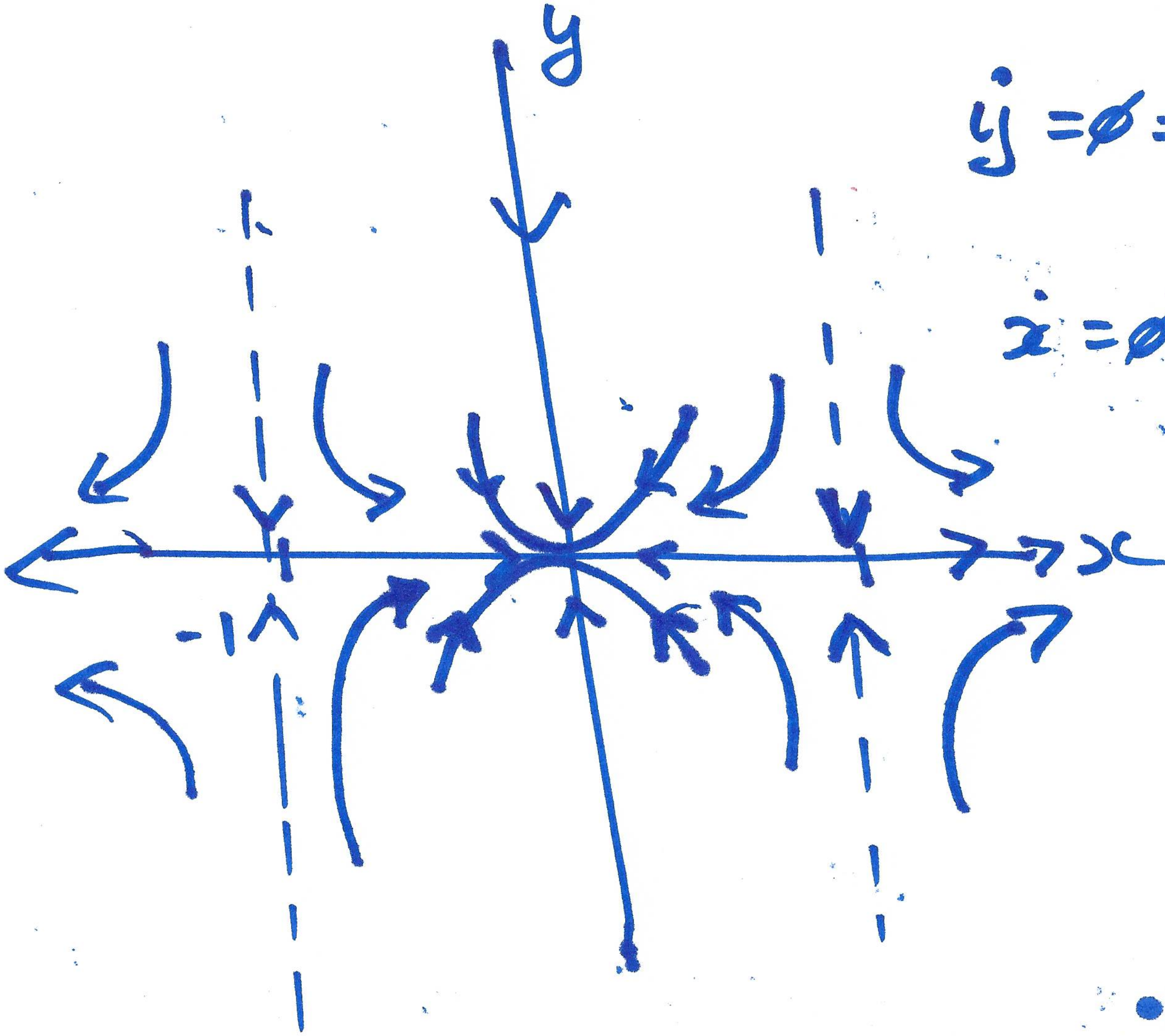
$(\pm 1, 0), \; (0, 0)$

$(\pm 1, 0) \quad J = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda_1 = 2, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -2, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$





$\dot{y} = \emptyset \Rightarrow y = \emptyset$

purely horizontal

$\dot{x} = \emptyset \Rightarrow x = \emptyset, \pm 1$

purely vertical

Example

$$\dot{x} = -y + \alpha x(x^2+y^2)$$
$$\dot{y} = x + \alpha y(x^2+y^2) \Rightarrow \begin{array}{l}\dot{\theta} = 1\\ \dot{r} = \alpha r^3\end{array}$$

α is a given parameter.

Find fixed points, calculate linearization around them, classify stability, show linearization does NOT capture the behavior of the system.

Sltn

$x = y = \emptyset$;

two

$$J = \begin{pmatrix} \alpha(x^2+y^2) + \alpha x^2 \cdot 2 & -1 + 2xy \\ 1 + 2\alpha xy & \alpha(x^2+y^2) + 2y^2\alpha \end{pmatrix}$$

$$J(\emptyset,\emptyset) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$Tr = \emptyset$, $\Delta = 1$

Polar coordinates, 15

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$ 16

$y = r\sin\theta$

$\theta = \mathrm{ArcTan}(y/x)$

$$\frac{d}{dx}\mathrm{ArcTan}\,x = \frac{1}{1+x^2}$$

$$\theta(t) = \mathrm{ArcTan}\left(\frac{y(t)}{x(t)}\right)$$

$$\frac{d}{dt}\theta(t) = \frac{1}{1+(y/x)^2}\cdot\left(\frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2}\right)\cdot\frac{x}{x}$$

$$= \frac{1}{1+(y/x)^2}\left[\frac{\dot{y}x}{x^2} - \frac{y\dot{x}}{x^2}\right] =$$

$$= \frac{1}{x^2+y^2}\left[\dot{y}x - y\dot{x}\right] = \frac{x^2+y^2}{x^2+y^2} = 1$$

$$-y\,\dot{x} = -y^2 + \alpha x y (x^2+y^2)$$

$$x\,\dot{y} = x x + x \alpha y (x^2+y^2)$$

$$\dot{y} x - y \dot{x} = x^2 + y^2$$

$$r = \sqrt{x^2+y^2}$$

$$r^2 = x^2 + y^2$$

$$\dot{r} r = \dot{x} x + y \dot{y} = \alpha r^4 \Rightarrow \boxed{\begin{aligned} \dot{r} &= \alpha r^3 \\ \dot{\theta} &= 1 \end{aligned}}$$

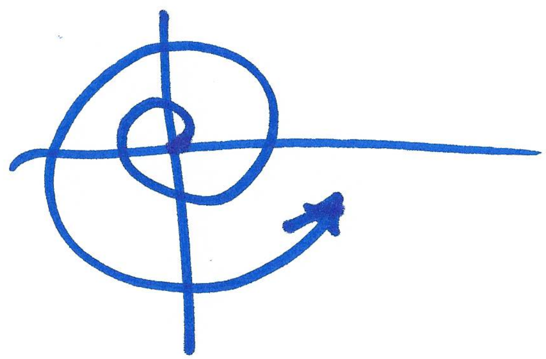
$$x\dot{x} = -y x + \alpha x^2 (x^2+y^2)$$

$$y\dot{y} = y x + \alpha y^2 (x^2+y^2)$$

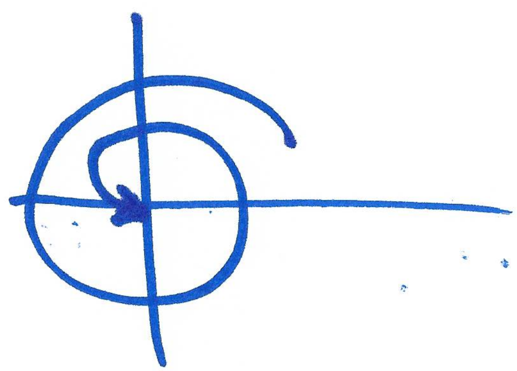
$$x\dot{x} + y\dot{y} = \alpha (x^2+y^2)^2 = \alpha r^4$$

17

$\alpha > 0$



$\alpha < 0$



Lotka-Voltera System

Rabbits and Sheep
(on an isolated land)

Rabbits $x(t)$

Sheep $y(t)$

Interactions

$$\frac{d}{dt}x(t) = x(t)\left(3 - x(t) - 2y(t)\right)$$

$$\frac{d}{dt}y(t) = y(t)\left(2 - y(t) - x(t)\right)$$

simple multiplication

LOGISTIC MODEL

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - y - x)$$

$(\emptyset, 2)$

$(3, \emptyset)$

$x = \emptyset \Rightarrow$

1st equation $\Rightarrow x = \emptyset \Rightarrow y = 2$ (second eqn)

2nd eq $y = \emptyset \Rightarrow$ first $x = 3$.

$x + 2y = 3$ $\quad y = 1$

$x + y = 2$ $\quad x = 1$

$(1, 1)$