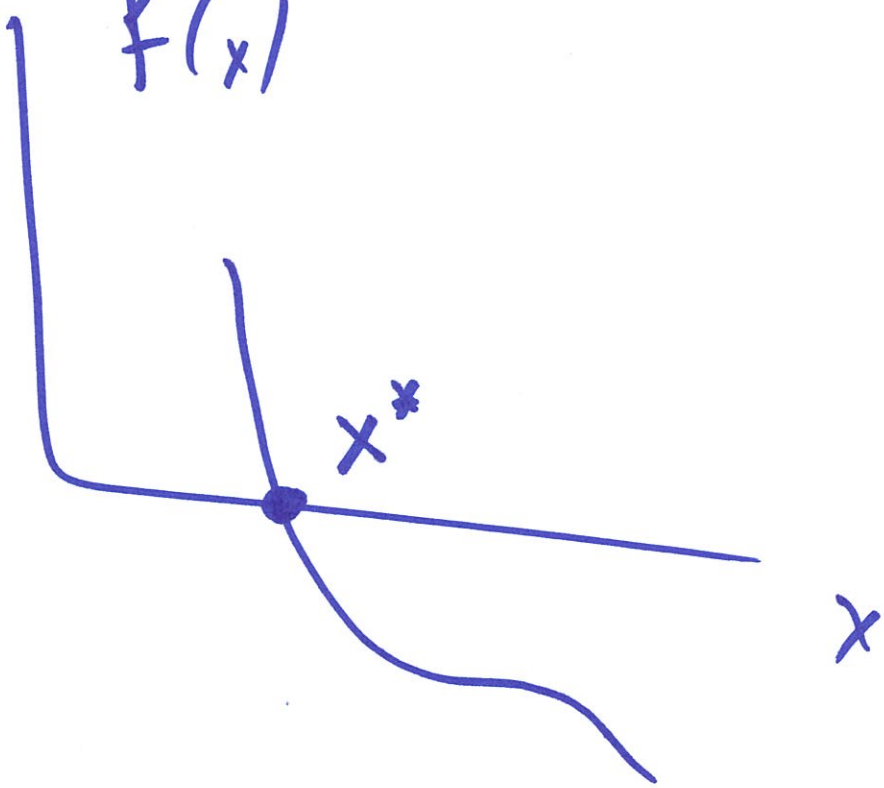
A



Find x^* by iterations

x_0

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$x_{k+1} = g(x_k)$

Want

$\lim_{k \to \infty} x_k = x^*$

B

$$x_{k+1} = g(x_k)$$

$$\epsilon_k = x_k - x^*$$

$$x_k = x^* + \epsilon_k$$

$$|g'(x^*)| < 1$$

$$x^* + \epsilon_{k+1} = \underbrace{g(x^* + \epsilon_k)}_{}$$

Taylor expansion

$$x^* + \epsilon_{k+1} = g(x^* + \epsilon_k) \approx$$

$$\approx g(x^*) + \epsilon_k \cdot g'(x^*)$$

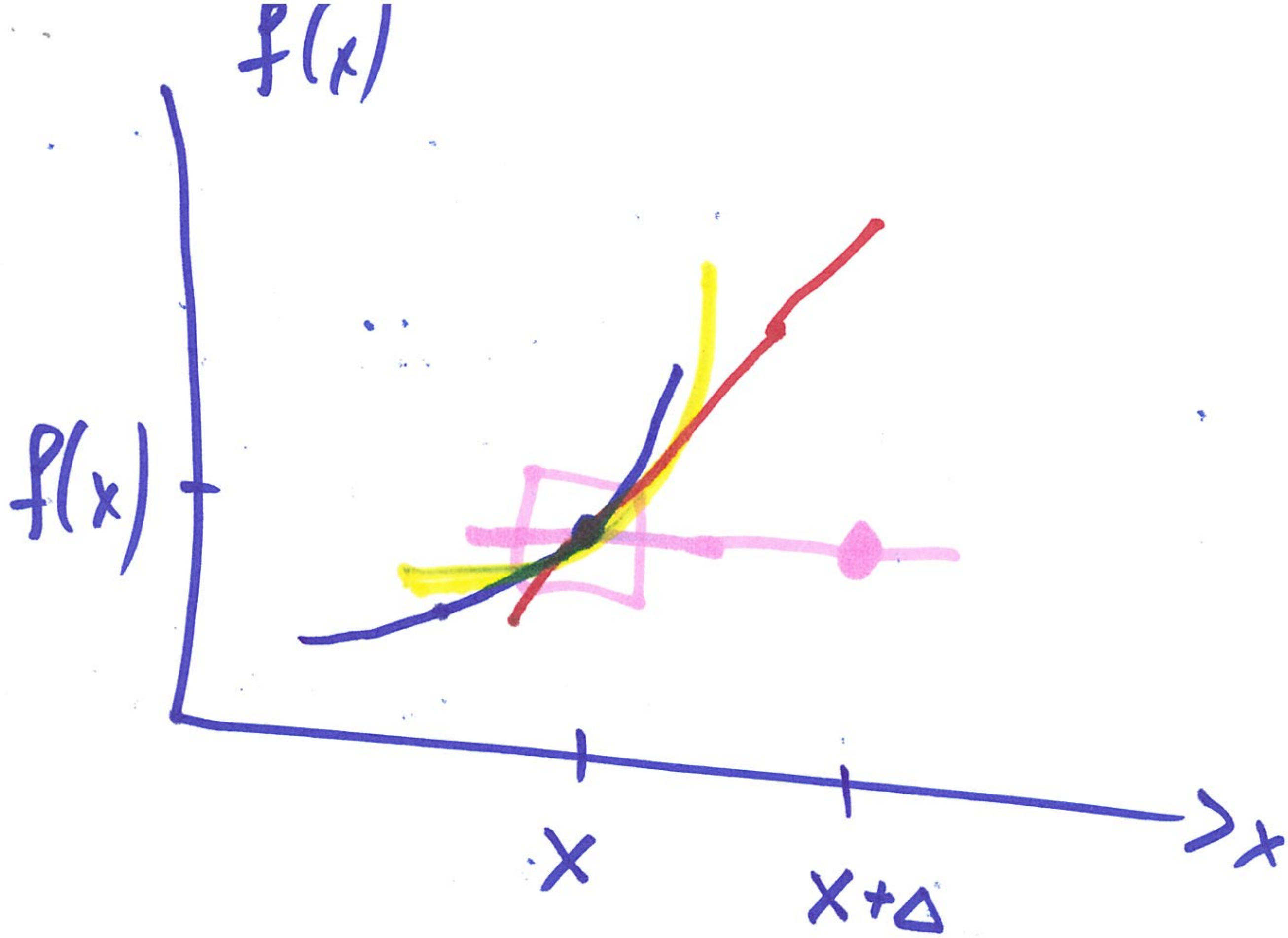
$$x^* = g(x^*)$$

$$\boxed{\epsilon_{k+1} = \epsilon_k \cdot g'(x^*)}$$

Linear convernce

$$\lim_{k \to \infty} \epsilon_k = 0 \quad \text{IF } \cancel{0}\, g'(x^*) < 1$$

C



$$f(x+\Delta) = f(x) + f'(x)\Delta + f''(x)\frac{\Delta^2}{2}$$

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

$$f(x+\Delta) = \sum_{k=0}^{\infty} \left(\frac{d}{dx}\right)^k f(x) \cdot \frac{\Delta^k}{k!}$$

$$k! = k \cdot (k-1)!$$

$$0! = 1$$

D

$$f(x+\Delta) = f(x) + \Delta f'(x) + \Delta^2 f''(x)/2 + f'''(x)\,\Delta^3/6 + f''''(x)\frac{\Delta^4}{24} + \ldots$$

E

$$\epsilon_{k+1} = \epsilon_k \cdot g'(x^*)$$

Best possible case

$$g'(x^*) = 0$$

$$x^* + \epsilon_{k+1} = g(x^* + \epsilon_k) \simeq$$

$$\simeq g(x^*) + g'(x^*) \cdot \epsilon_k + g''(x^*)\,\epsilon_k^2/2 + \dots$$

IF $g'(x^*) = 0$ and

$g''(x^*) \neq 0$

Then $\boxed{\epsilon_{k+1} = g''(x^*)\,\epsilon_k^2/2}$

quadratic convergence

$\epsilon_{k+1} = C \cdot \epsilon_k^r$

r=1 LINEAR - BISECTION

r=2 - quadratic

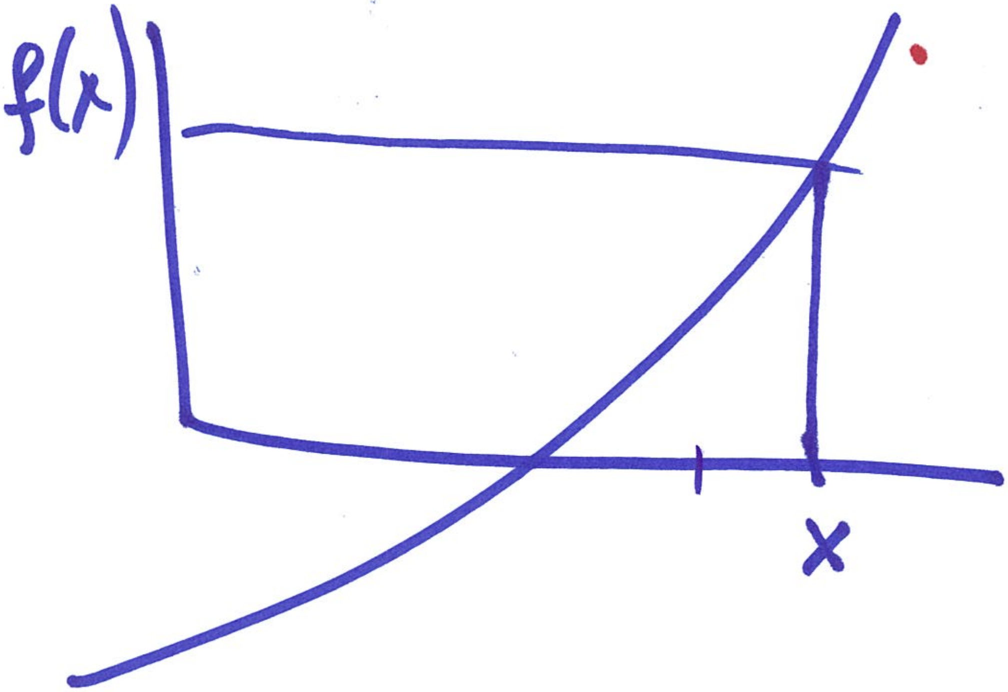
$\Updownarrow$

$\epsilon_{k+1} = g'(x^*) \cdot \epsilon_k$

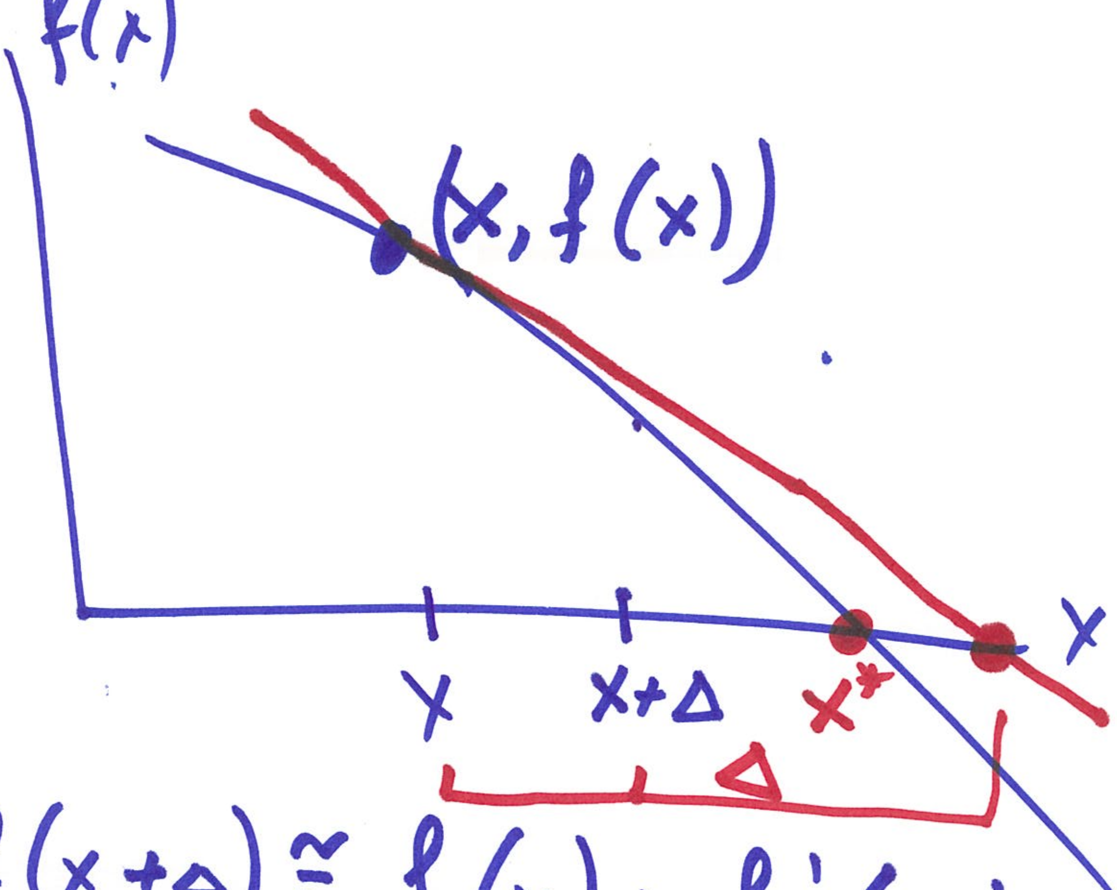
$g'(x^*) \neq 0$

$C = g'(x^*)$

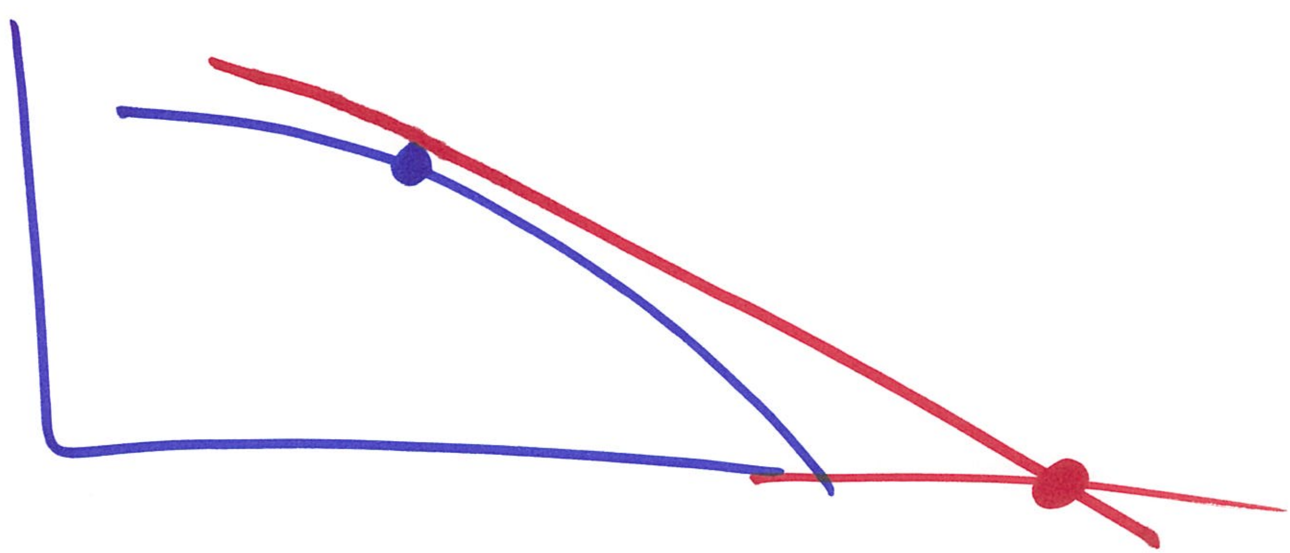
Newton's method



H



$$f(x+\Delta) \cong f(x) + f'(x)\cdot\Delta = \emptyset$$



I

x_k

$$f(x_{k+1}) = f(x_k + \overbrace{x_{k+1} - x_k}^{\Delta}) =$$

$$\simeq f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$$

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$$

$\cancel{f'(x_k) =}$

$$f'(x_k)(x_{k+1} - x_k) = - f(x_k)$$

$$x_{k+1} - x_k = - \frac{f(x_k)}{f'(x_k)}$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Newton's method

J

$f(x) = x^2 - 1, \quad x > \emptyset$

Forget $x^* = 1$

$f'(x) = 2x$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} =$$

$$= x_k - \frac{x_k^2 - 1}{2x_k} =$$

$$= x_k - \frac{x_k}{2} + \frac{1}{2x_k} =$$

$$= \frac{1}{2}\left(x_k + 1/x_k\right) = x_{k+1}$$

$x_0 = 10$

$$x_{k+1} = \frac{1}{2}\left(x_k + 1/x_k\right)$$ K

$$x_0 = 10$$

$$x_1 = \frac{1}{2}\left(10 + 1/10\right) = \frac{1}{2}\cdot 10.1 = 5.05$$

$$= 5 + 1/20 = \frac{101}{20}$$

$$x_2 = \frac{1}{2}\left(\frac{101}{20} + \frac{20}{101}\right)$$

$$= \frac{1}{2}\left(\frac{10201 + 400}{2020}\right)$$

$$= \frac{1}{2}$$

L

Example

$f(x) = x^2;$

$f'(x) = 2x;$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} =$$

$$= x_k - \frac{x_k^2}{2x_k} = \frac{x_k}{2}$$

$x^* = \emptyset,$ $\epsilon_k = x_k$

$$\epsilon_{k+1} = \epsilon_{k+1}$$

$$\epsilon_{k+1} = \frac{\epsilon_k}{2}$$

M

$$x_{k+1} = \frac{1}{2}\left(x_k + 1/x_k\right)$$

$$x^* = 1$$

$$x_k = 1 + \epsilon_k$$

$$\epsilon_{k+1} + 1 = \frac{1}{2}\left(1 + \epsilon_k + \frac{1}{1+\epsilon_k}\right)$$

$$= \frac{1}{2}\left(1 + \epsilon_k + 1 - \epsilon_k + \epsilon_k^2/2\right)$$

$$= \frac{1}{2}\left(2 + \epsilon_k^2/2\right) = 1 + \epsilon_k^2/4$$

$$\epsilon_{k+1} = \epsilon_k^2/4$$

N

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = g(x_k)$$

Newton's method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x^*) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right)\Bigg|_{x=x^*} =$$

$$= \left(1 - \frac{f'(x)}{f'(x)} + \frac{f(x)\,f''(x)}{(f'(x))^2}\right)\Bigg|_{x=x^*}$$

$$= 1 - 1 + \frac{f(x)\,f''(x)}{(f'(x))^2} =$$

$$\cancel{= \left(\frac{f(x)}{f'(x)}\right)_{x=x^*} = \frac{f(x^*)}{f'(x^*)}}$$

0

$$g'(x) = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$$g'(x^*) = \frac{f(x^*)\, f''(x^*)}{(f'(x^*))^2}$$

$f(x^*) = \emptyset,$

$g'(x^*) = \emptyset$ if $f'(x^*) \neq 0$

Newton method has quadratic convergence if $f'(x^*) \neq 0$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

P

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$x_k = x^* + \epsilon_k$$

$$x_{k+1} = x^* + \epsilon_{k+1}$$

$$x^* + \epsilon_{k+1} = x^* + \epsilon_k - \frac{f(x^* + \epsilon_k)}{f'(x^* + \epsilon_k)}$$

$$= x^* + \epsilon_k - \frac{f(x^*) + f'(x^*)\epsilon_k + f''(x^*)\epsilon_k^2/2}{f'(x^*) + f''(x^*)\epsilon_k}$$

$$= x^* + \epsilon_k - \frac{f(x^*)}{f'(x^*)} + \epsilon_k + \frac{f''(x^*)\epsilon_k^2}{2f'(x^*)},$$

$$f(x^*) = 0$$

$$\boxed{\epsilon_{k+1} = \frac{f''(x^*)}{2f'(x^*)} \cdot \epsilon_k^2 + \dots}$$

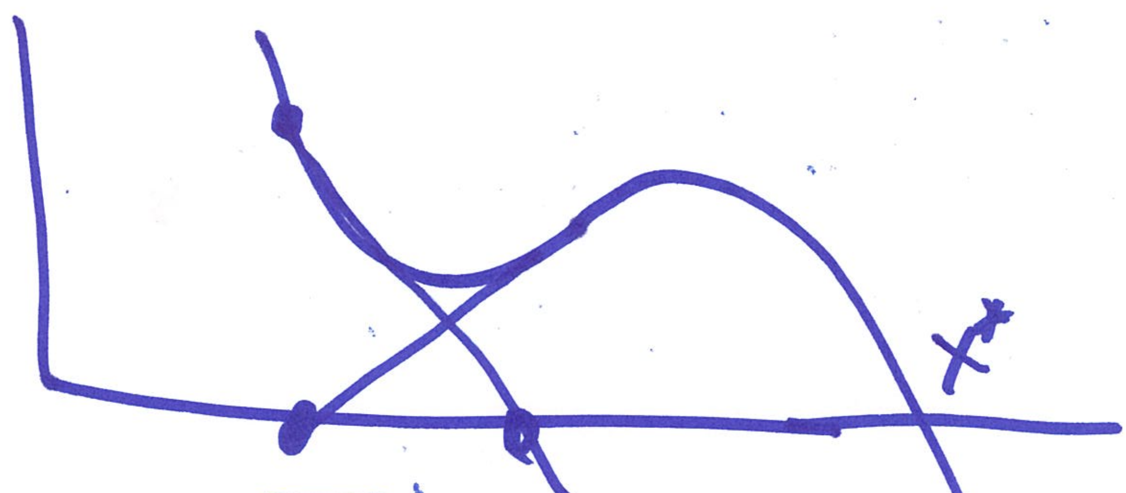
Q

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

quadratic

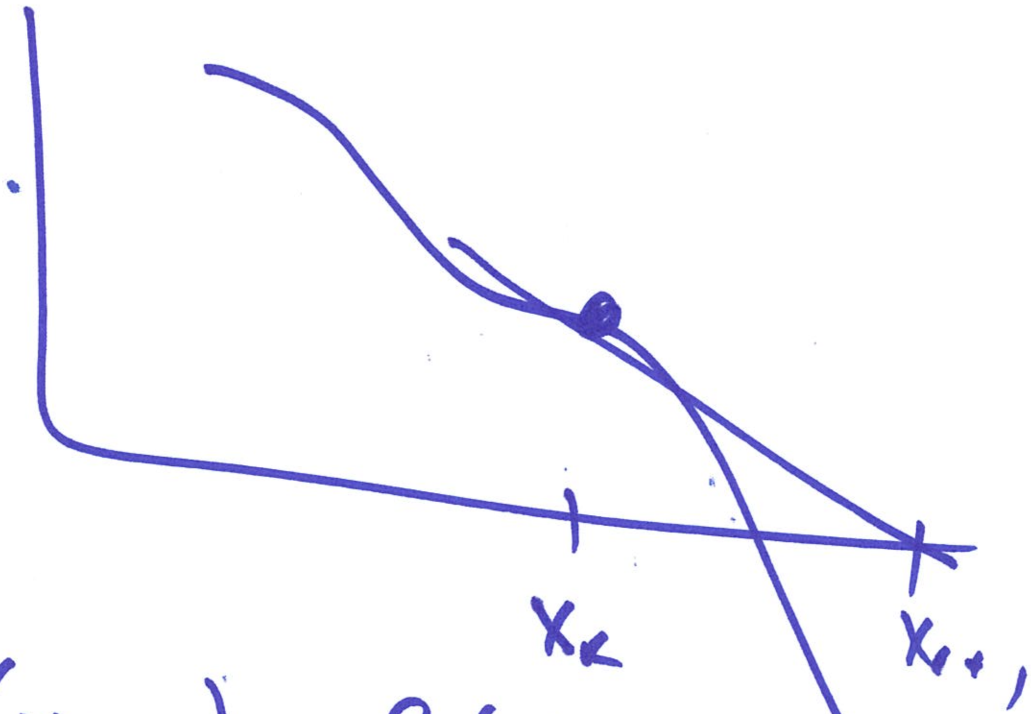
convergence if

$f'(x^*) \neq 0$



$\overline{f'(x)}$ need to know analytically

12



$$f(x+\Delta) = f(x) + f'(x)\,\Delta + f''(x)\frac{\Delta^2}{2}$$