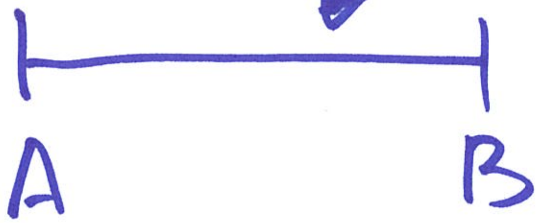
#

$$I = \int_A^B f(x)\,dx$$

N equal parts



$$h = \frac{B-A}{N}$$

↓

spacing

$$x_i = A + i\,h = A + i\,\frac{B-A}{N}$$

$$x_0 = A; \quad x_N = B$$

$$I = \int_A^B f(x)\,dx =$$

$$= \int_{x_0}^{x_1} dx\, f(x) + \left(\int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \dots + \int_{x_{N-1}}^{x_N} \right) dx\, f(x)$$

$$\int_A^B f(x)\,dx = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} f(x)\,dx$$

$$\int_{x_r}^{x_{r+1}} dx\, f(x) = ?$$

$x_r \equiv a$

$x_{r+1} \equiv b$ $\qquad m \equiv \dfrac{a+b}{2}$

$$\int_a^b f(x)\,dx = \begin{cases} \text{Midpoint} & f(m)(b-a) \\ \text{Trap} & (f(a)+f(b))(b-a)/2 \\ \text{Simps} & \dfrac{b-a}{6}\left(f(a)+4f(m)+f(b)\right) \end{cases}$$

Method of undetermined coefficients

B

- $\int_a^b f(x)dx = \omega \cdot f(m)$
  ↳ weight

→ $f(x) = 1$;

$$\int_a^b 1 \cdot dx = \boxed{b - a = \omega \cdot}$$

- $\int_a^b f(x)dx = \omega_1 f(a) + \omega_2 f(b)$

→ $f(x) = 1$: $\boxed{b - a = \omega_1 + \omega_2}$

→ $f(x) = x$; $\int_a^b x dx = \frac{b^2 - a^2}{2} =$

$$= \boxed{\frac{(b-a)(b+a)}{2} = \omega_1 \cdot a + \omega_2 \cdot b}$$

$\omega_1 = \omega_2 = \frac{b - a}{}$

$$\int_a^b f(x)dx = \omega_1 f(a) + \omega_2 f(m) + + \omega_3 f(b)$$

$\to f(x)=1$: $b-a = \omega_1 + \omega_2 + \omega_3$

$\to f(x)=x$; $\dfrac{b^2-a^2}{2} = \omega_1 a + \omega_2 m + \omega_3 b$

$\to f(x)=x^2$; $\displaystyle\int_a^b x^2 dx = \frac{b^3-a^3}{3} = \omega_1 a^2 + \omega_2 m^2 + \omega_3 b^2$

$$\omega_1 = \omega_3 = \frac{b-a}{6}$$

$$\omega_2 = \frac{2}{3}(b-a) \qquad m = \frac{a+b}{2}$$

D

Gauss guadratures:

$$\int_{-1}^{1} f(x)\,dx = \omega_1 \cdot f(z_1) + \omega_2 f(z_2)$$

nodes, variables

$$f(x)=1: \int_{-1}^{1} dx = 2 = \omega_1 + \omega_2$$

$$f(x)=x; \int_{-1}^{1} x\,dx = \emptyset = \omega_1 z_1 + \omega_2 z_2$$

$$f(x)=x^2; \int_{-1}^{1} x^2\,dx = \frac{2}{3} = \omega_1 z_1^2 + \omega_2 z_2^2$$

$$f(x)=x^3; \int_{-1}^{1} x^3\,dv = \emptyset = \omega_1 z_1^3 + \omega_2 z_2^3$$

$\omega_1 = \omega_2$

$z_1 = -z_2$

$\omega_1 = \omega_2 = 1$

$$z_2 = -z_1 = \frac{1}{\sqrt{3}}$$

ε

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(z_1) + \omega_2 f(z_2) + \omega_3 f(z_3)$$

1 $\quad 2 = \omega_1 + \omega_2 + \omega_3$

x $\quad \emptyset = \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3$ $\checkmark$

x^2 $\quad \frac{2}{3} = \omega_1 z_1^2 + \omega_2 z_2^2 + \omega_3 z_3^2$

x^3 $\quad \emptyset = \omega_1 z_1^3 + \omega_2 z_2^3 + \omega_3 z_3^3$ $\checkmark$

x^4 $\quad \frac{2}{5} = \omega_1 z_1^4 + \omega_2 z_2^4 + \omega_3 z_3^4$

x^5 $\quad \emptyset = \omega_1 z_1^5 + \omega_2 z_2^5 + \omega_3 z_3^5$ $\checkmark$

$\omega_1 = \omega_3;\ z_2 = \emptyset;\ z_1 = -z_3$

$$2 = 2\omega_1 + \omega_2$$

$$\frac{2}{3} = 2\omega_1 z_1^2$$

$$\frac{2}{5} = 2\omega_1 z_1^4$$

$$z_1^2 = \frac{2 \cdot 3}{5 \cdot 2} = \frac{3}{5}$$

$$z_1 = \sqrt{3/5} = -z_3$$

ɔ+c

$$w_1 = \frac{1}{3 \cdot z_1^2} = \frac{1 \cdot 5}{3 \cdot 3} = \frac{5}{9}$$

$$w_2 = 2 - 2w_1 = 2 - \frac{10}{9} = \frac{8}{9}$$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left(f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right) + \frac{8}{9} f(0)$$

$$\int_a^b f(x)dx \neq \int_{-1}^{1} f(x)dx$$

$$\int_a^b f(x)dx = A\int_{-1}^{1} f(z)dz$$

$x = \alpha z + \beta$: If $x = a$, $z = -1$

$dx = \alpha\, dz$ $\quad$ $x = b$, $z = 1$

$$a = -\alpha + \beta \qquad \beta = \frac{a+b}{2}$$

$$b = \alpha + \beta \qquad \alpha = \frac{b-a}{2}$$

$$x = \frac{b-a}{2} z + \frac{a+b}{2}$$

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2} z + \frac{a+b}{2}\right)dz$$

$$= \frac{b-a}{2}\left(f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{2}}\right) + f\left(\frac{a+b}{2} - \frac{b-a}{\sqrt{2}}\right)\right)$$

$$\Big[\, f(x) = f(m + x - m) =$$

$$= f(m) + \sum_{k=1}^{\infty} \left(\frac{d}{dx}\right)^k f(m) \cdot \frac{(x-m)^k}{k!}$$

$$= f(m) + f'(m)(x-m) + f''(m)(x-m)^2/2 + f'''(m)(x-m)^3/6 + f''''(m)(x-m)^4/24 \,\Big]$$

$$\int_a^b [\;]$$

$$\int_a^b f(x)\,dx = \overset{\text{I}}{\int_a^b f(m)\,dx} + \overset{\text{II} = 0}{\int_a^b f'(m)(x-m)\,dx}$$

$$+ \overset{\text{III}}{\int_a^b f''(m)(x-m)^2/2\,dx}$$

$$+ \overset{\text{IV} = 0}{\int_a^b f'''(m)(x-m)^3/6\,dx}$$

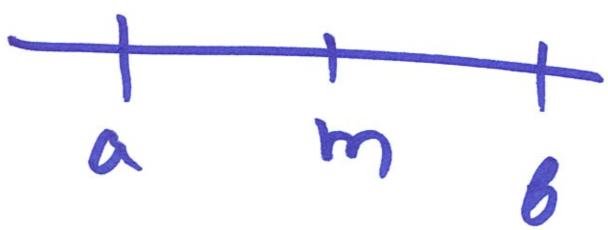
$$+ \int_a^b f''''(m)(x-m)^4/24\,dx \;\; \text{V}$$

$\textcircled{I} = \int_a^b f(m)\,dx = (b-a) f(m)$

$\mathrm{II} = \int_a^b f'(m)(x-m)\,dx =$

$= f'(m) \int_a^b (x-m)\,dx = f'(m) \left. \frac{(x-m)^2}{2} \right|_{x=a}^{x=b} =$

$= f'(m) \frac{(b-m)^2 - (a-m)^2}{2} = \emptyset$



$(b-m) = (m-a) =$

$= -(a-m)$

$\textcircled{III} = \int_a^b f'''(m) \frac{(x-m)^3}{6}\,dx$

$= f'''(m) \frac{1}{6} \cdot \left[(a-m)^4/4 - (b-m)^4/4 \right] = \emptyset$

1

$$\mathrm{III} = \int_a^b f''(m)\frac{(x-m)^2}{2}\,dx =$$

$$= f''(m)\,\frac{(x-m)^3}{2\cdot 3}\Bigg|_{x=a}^{x=b} =$$

$$= \frac{f''(m)}{6}\left[(b-m)^3 - (a-m)^3\right]$$

$$= \frac{f''(m)}{6}\,\frac{(b-a)^3}{2^3}\cdot 2 = \frac{f''(m)(b-a)^3}{24}$$

$$b - m = \frac{b-a}{2}$$

$$a - m = -\frac{b-a}{2}$$

I·V $=$ M

$$= \int_a^b f''''(m)(x-m)^4/24\,dx$$

$$= f''''(m)\cdot\frac{1}{24}\cdot\frac{(x-m)^5}{5}\Bigg|_{x=a}^{x=b} =$$

$$= f''''(m)\frac{1}{24}\cdot\frac{1}{5}\cdot\frac{2}{2^5}(b-a)^5$$

$$= \frac{(b-a)^5}{1920} f''''(m) \qquad 24\cdot5\cdot2^5=1920$$

H

$$\int_a^b f(x)\,dx = (b-a)\,f(m) + \frac{1}{24} f''(m)(b-a)^3 + \frac{1}{1920} f''''(m)(b-a)^5$$

$$\int_A^B f(x)\,dx = \sum_{k=\emptyset}^{N-1} f\left(\frac{x_k + x_{k+1}}{2}\right) \cdot \frac{B-A}{N}$$

$$E = \left| \int_A^B f(x)\,dx - \sum_{r=0}^{N-1} f\left(\frac{x_r + x_{r+1}}{2}\right) \frac{B-A}{N} \right|$$

0

$$= \left| \sum_{r=0}^{N-1} f''(\eta_i) \cdot \left(\frac{B-A}{N}\right)^3 \cdot \frac{1}{24} \right| \qquad A \leq \cancel{x \leq} B$$

$$x_r \leq \eta_i \leq x_{r+1}$$

$$\leq \sum_{r=0}^{N-1} |f''(\eta_i)| \frac{(B-A)^3}{24 \cdot N^3} \qquad \max_{A \leq x \leq B} |f''(x)| = \|f(x)\|_\infty$$

$$\leq \sum_{r=0}^{N-1} \|f''\|_\infty \cdot \frac{1}{24} \cdot h^3 = \qquad h = \frac{B-A}{N}$$

$$= N \cdot h \cdot h^2 \|f''\|_\infty \cdot \frac{1}{24} =$$

$$= (B-A)\, h^2 \|f''\|_\infty \cdot \frac{1}{24}$$

Theorem 6.1

IF $f(x) \in C^2[A, B]$ [C^2 is twice continuously differentiable] function, then the composit midpoint method has an error of

$$\left| \int_A^B f(x)\,dx - \sum_{r=0}^{N-1} f\left(\frac{x_r + x_{r+1}}{2}\right) h \right| \le \frac{1}{24} \|f''\|_\infty h^2 (B-A).$$

n

$$\int_a^b f(x)\,dx =$$

$$= \frac{(b-a)}{2}\left(f(a) + f(b)\right)$$

$$+ f''(m)(b-a)^3\left(\frac{1}{24} - \frac{1}{4\times 2}\right) \rightarrow -\frac{1}{12}$$

$$+ f''''(m)(b-a)^5\left(\frac{1}{1920} - \frac{1}{24\times 16}\right) \rightarrow -\frac{1}{480}$$

$$\int_a^b f(x)\,dx =$$

$$= \frac{(b-a)}{2}\left(f(a) + f(b)\right)$$

$$- \frac{1}{12} f''(m)(b-a)^3 - \frac{1}{480} f''''(m)(b-a)^5$$

0

$$f(a) = f(m) + f'(m)(a-m) + f''(m)(a-m)^2/2 + f'''(m)(a-m)^3/6 + f''''(m)(a-m)^4/24$$

$$f(b) = f(m) + f'(m)(b-m) + f''(m)(b-m)^2/2 + f'''(m)(b-m)^3/6 + \frac{f''''(m)}{24}(b-m)^4$$

$$f(a) + f(b) = 2f(m) + f''(m)(a-m)^2 + \frac{f''''(m)}{\cancel{24}} \frac{(a-m)^4}{12}$$

$$(a-m)^2 = \frac{(b-a)^2}{4} \; ; \quad (a-m)^4 = \frac{(b-a)^4}{2^4} = \frac{(b-a)^4}{16}$$

$$f(m) = \frac{f(a)+f(b)}{2} - \frac{f''(m)(b-a)^2}{4 \times 2} - \frac{f''''(m)(b-a)^4}{2 \times 12 \cdot 16}$$

a

$$f(a) = f(m+a-m) = f(m) + f'(m)(a-m) + \frac{f''(m)(a-m)^2}{2} + \frac{f'''(m)(a-m)^3}{6} + \dots$$