Interpolation

A

$(x_i, y_i) \quad i = 1..n$

$\varphi_i(x)$;

Find n values a_i

such that

$$g(x) = \sum_{k=1}^{n} a_k \varphi_k(x) \text{ and}$$

$$g(x_i) = y_i, \quad i = 1..n$$

monomial: $\varphi_i(x) = x^{i-1}$

Lagrange: $$\varphi_i(x) = \frac{\prod_{\substack{l=1 \\ l \neq i}}^{n} (x - x_l)}{\prod_{\substack{l=1 \\ l \neq j}}^{n} (x_i - x_l)}$$

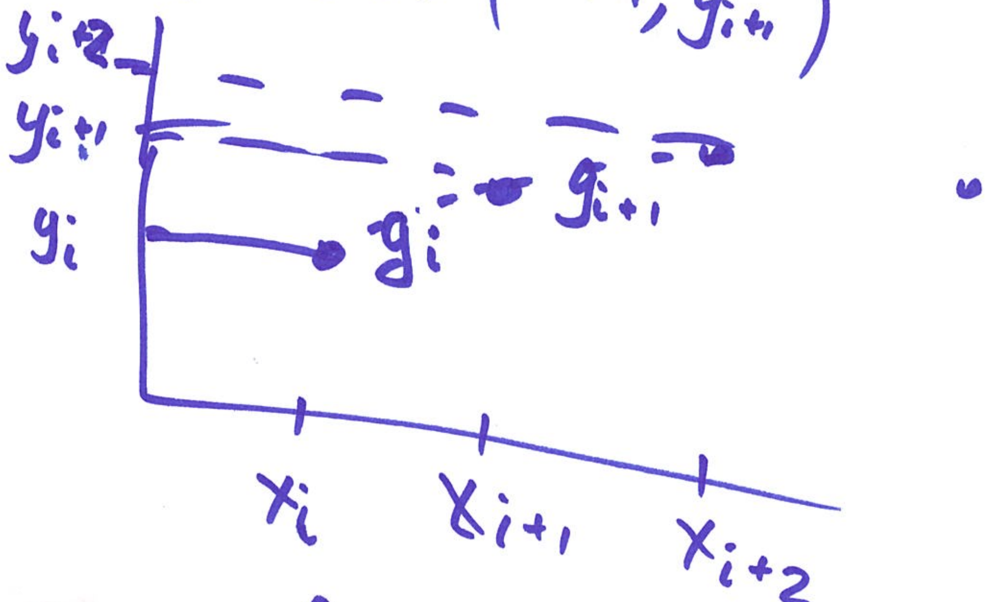
$$\ell_i(x_j) = \delta_j^i$$

PW linear

B

$$g_i(x_i) = y_i + \frac{y_{i+1}-y_i}{x_{i+1}-x_i}\cdot(x-x_i)$$

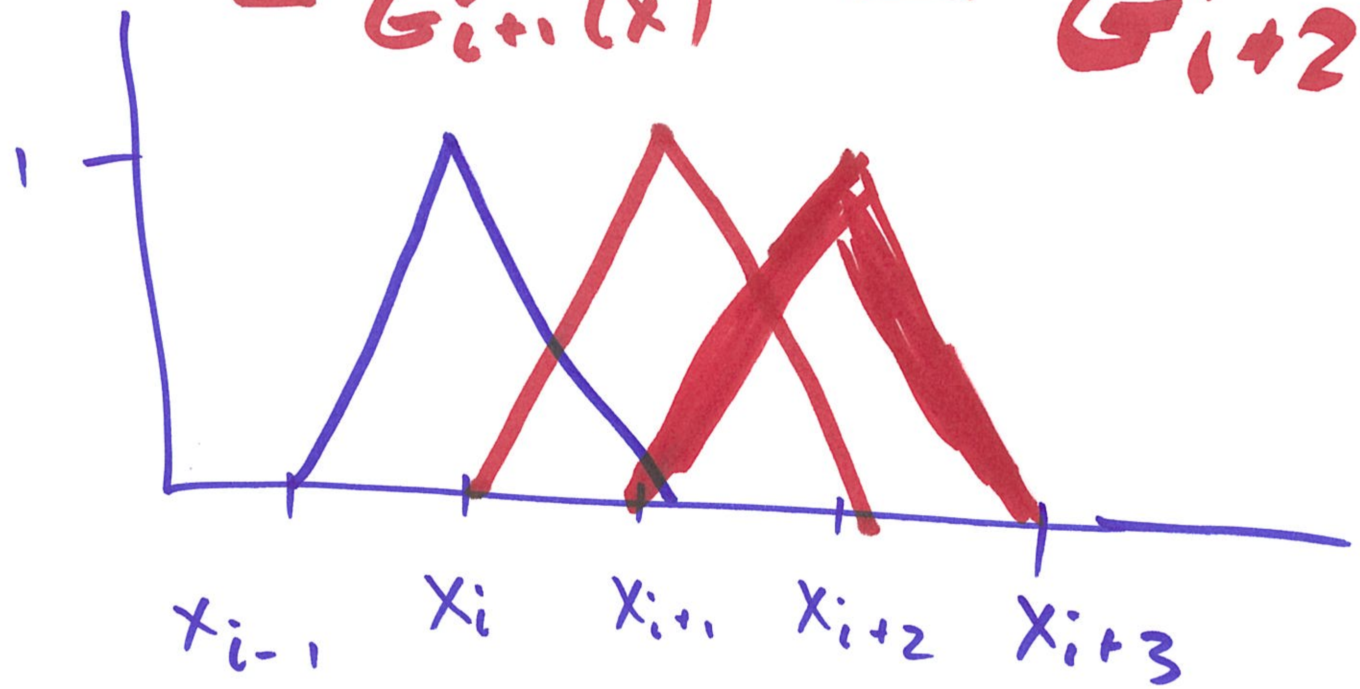
$$(x_i, y_i);\ (x_{i+1}, y_{i+1})$$



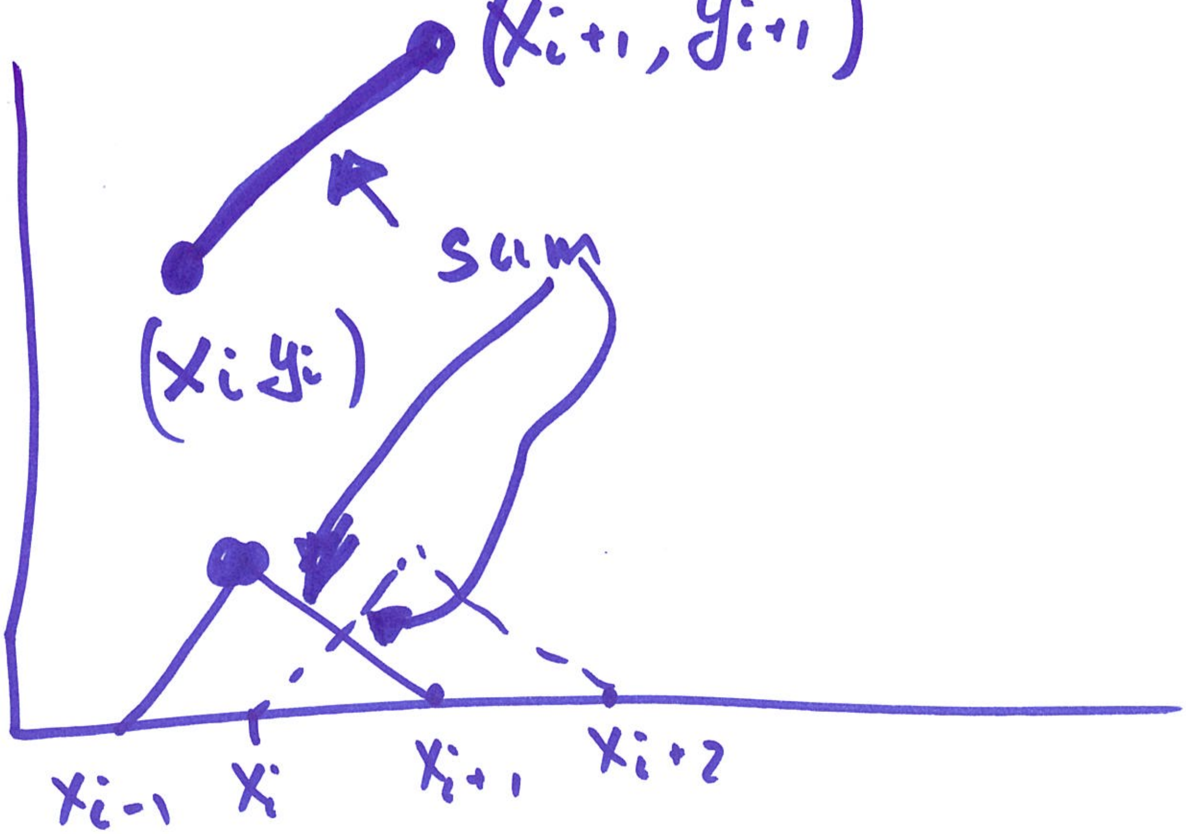
$$G_i(x) = \begin{cases} \emptyset, & x \le x_{i-1} \\ \dfrac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x-x_{i+1}}{x_i-x_{i+1}} & x_i \le x \le x_{i+1} \\ \emptyset & x > x_{i+1} \end{cases}$$

$$\boxed{G_i(x_j)=\delta_j^i}$$

— $G_i(x)$
— $G_{i+1}(x)$ — $G_{i+2}(x)$



$$f(x) = \sum_{k=1}^{n} y_k \, G_k(x)$$



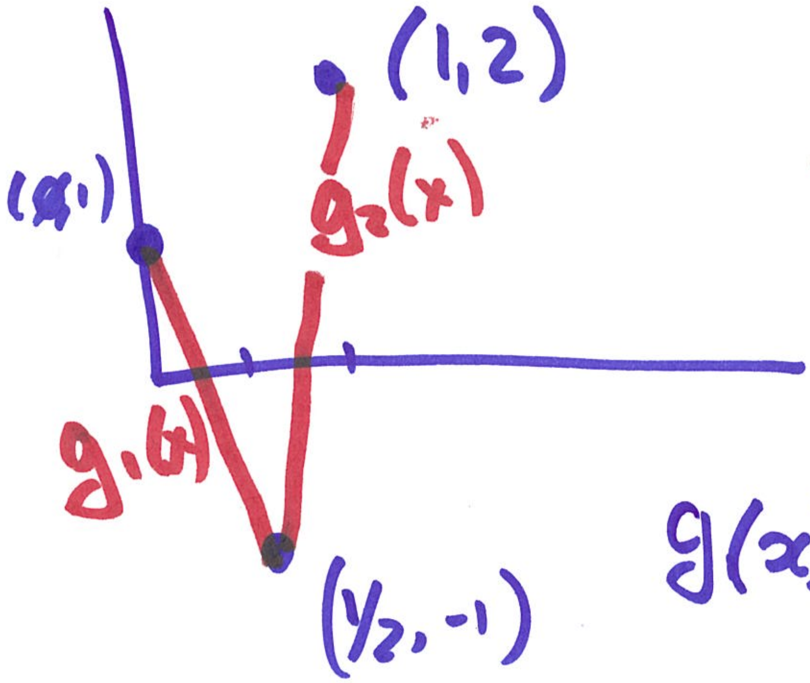
x	$\emptyset$	$1/2$	1
y	1	-1	2

D

$(0,1) = (x_1, y_1)$

$(1/2, -1) = (x_2\, y_2)$

$(1,2) = (x_3\, y_3)$



$$g(x) = \begin{cases} g_1(x), \emptyset \le x \le 1/2 \\ g_2(x) \quad 1/2 \le x \le 1 \end{cases}$$

$$g_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$= 1 + \frac{-1-1}{1/2 - \emptyset}\, x =$$

$$= 1 - 4x$$

E

$$g_2(x) = y_2 + \frac{y_3 - y_2}{x_3 - x_2}(x - x_2) \qquad 1$$

$$\cancel{= 2+}$$

$$= -1 + \frac{2-(-1)}{1-1/2}(x - 1/2) \qquad 2$$

$$= -1 + 6(x - 1/2) \qquad 3$$

$$= -4 + 6x \qquad 4$$

$$g_1(x) = 1 - 4x$$

$$g_2(x) = -4 + 6x$$

$x_0 = -1/2$ F

$(x_1, y_1) = (0, 1)$

$(x_2\ y_2) = (1/2, -1)$

$(x_3\ y_3) = (1, 2)$

$x_4 = 1.5$

$$G_1(x) = \begin{cases} (1)\ \ 0, & x \le x_0 \\ (2)\ \ \dfrac{x - x_0}{x_1 - x_0} & x_0 < x \le x_1 \\ (3)\ \ \dfrac{x - x_2}{x_1 - x_2} & x_1 \le x \le x_2 \\ (4)\ \ 0 & x > x_2 \end{cases}$$

(1) 0

$\dfrac{x + 1/2}{+1/2} = 2x + 1$

$\dfrac{x - 1/2}{-1/2} = -2x + 1$

0

$$G_1(x) = \begin{cases} 0, & x < -1/2 \\ 1 + 2x, & -1/2 \le x \le 0 \\ -2x + 1 & 0 \le x \le 1/2 \\ 0 & x > 1/2 \end{cases}$$

G

$$G_2(x) = \begin{cases} 0, & x < x_1 = 0 \\ \dfrac{x - x_1}{x_2 - x_3} & x_1 \le x \le x_2 \\ \dfrac{x - x_2}{x_2 - x_3} & x_2 \le x \le x_3 \\ 0 & x \ge x_3 \end{cases}$$

$$G_2(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{+1/2} = +2x & 0 \le x \le 1/2 \\ \dfrac{x - 1}{-1/2} = -2x + 2 & 1/2 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

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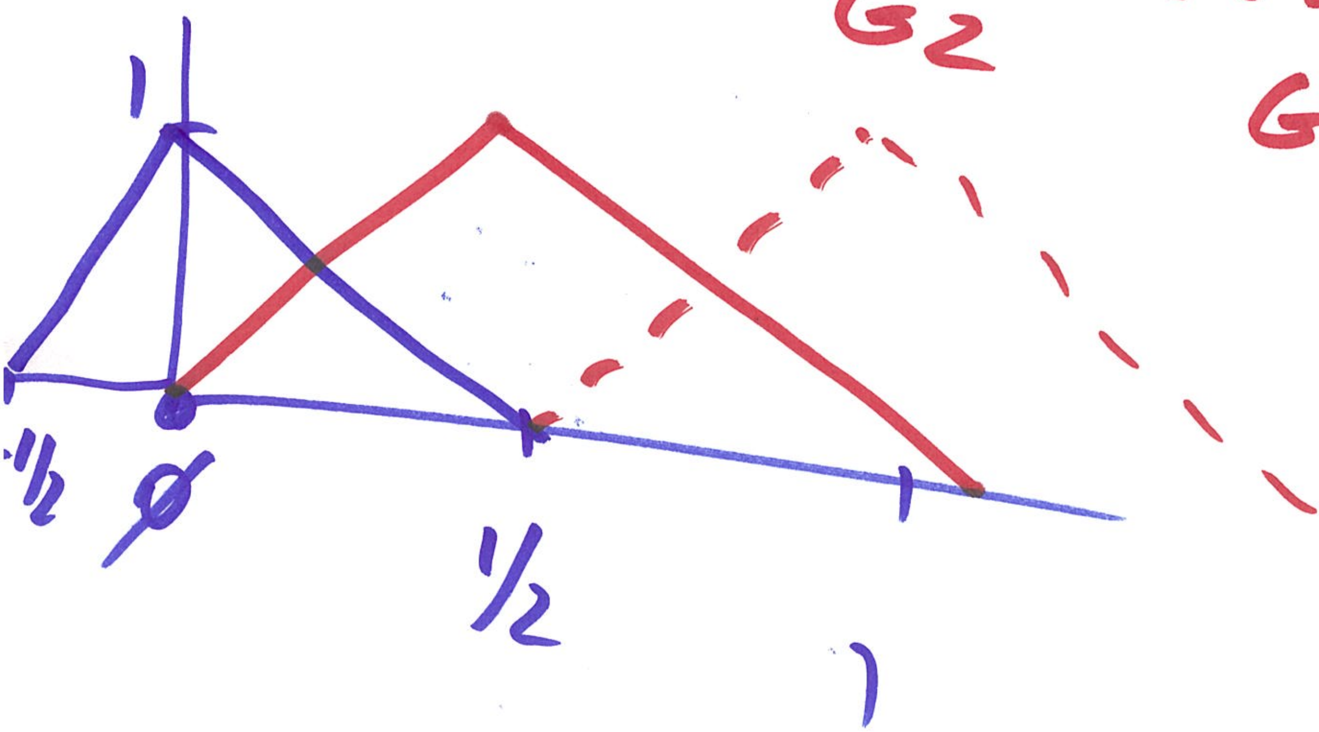
$$G_3(x) = \begin{cases} -1+2x ; \frac{1}{2} \leq x \leq 1 \\ 3-2x : 1 \leq x \leq \frac{3}{2} \\ \emptyset \quad \text{otherwise} \end{cases}$$

— G_1
— G_2
- - - G_3



Spline

is a pw polynomial of degree k which has $k-1$ continuous

derivatives.

Cubic Splines

are pw cubic polynomials with continuous first and second derivatives

$(x_i\ y_i)\quad i = 1..n$

$n-1$ cubic splines

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$1 \quad S_1 \quad 2 \quad S_2 \quad 3 \quad S_3 \quad 4$

2 exterior points

(x_1, y_1)

(x_n, y_n)

~~$y_1 = a_1 + b_1(x - x_1) + c_1($~~

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$

$$S_1(x_1) = \boxed{a_1 = y_1}$$

K

$$S_{n-1}(x_n) = y_n = a_{n-1} + b_{n-1}(x_n - x_{n-1}) + c_{n-1}(x_n - x_{n-1})^2 + d_{n-1}(x_n - x_{n-1})^3$$

Second equation

(x_l, y_l), $l = 2, 3 \ldots n-1$
interior points

$$\cancel{S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^2}$$
$$\cancel{S_2(x) = a_2 + b_2(x-x_2) + c_2(x-x_2)^2 + d_2(x-x_2)^2}$$

$$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3$$
$$S_2(x) = a_2 + b_2(x-x_2) + c_2(x-x_2)^2 + d_2(x-x_2)^3$$

$$S_1(x_2) = y_2$$
$$S_2(x_2) = y_2$$

$$y_2 = b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3$$
$$y_2 = a_2$$

$l = 2, 3 \ldots n-1$

$S_{l-1}(x) = a_{l-1} + b_{l-1}(x - x_{l-1}) \cancel{+ c_{l-1}(x - x_{l-1})}$
$+ c_{l-1}(x - x_{l-1})^2 + d_{l-1}(x - x_{l-1})^3$

$S_l(x) = a_l + b_l(x - x_l) + c_l(x - x_l)^2 + d_l(x - x_l)^3$

$S_{l-1}(x_l) = y_l = a_{l-1} + b_{l-1}(x_l - x_{l-1}) + c_{l-1}(x_l - x_{l-1})^2 + d_{l-1}(x_l - x_{l-1})^3$

$S_l(x_l) = y_l$

$= y_l = a_l$

2 (exterior points)

$+ 2 \cdot (n-2)$

M

$$S_e'(x) = b_e + 2c_e(x - x_e) + 3d_e(x - x_e)^2$$

$e-1$

$$S_{e-1}'(x) = b_{e-1} + 2c_{e-1}(x - x_e) + 3d_{e-1}(x - x_{e-1})^2$$

$$S_e''(x) = 2c_e + 6d_e(x - x_e)$$

$$S_{e-1}''(x) = 2c_{e-1} + 6d_{e-1}(x - x_{e-1})$$

First derivative is continious at the interior point:

$$S_{e-1}'(x_e) = S_e'(x_e)$$

2nd derivative continious

$$S_{e-1}''(x_e) = S_e''(x_e)$$

$$b_e + \quad = b_{e-1} + 2c_{e-1}(x_e - x_{e-1}) + 3d_{e-1}(x_e - x_{e-1})^2$$

N

$$S''_{\ell-1}(x_\ell) = S''_\ell(x_\ell)$$

$$2c_{\ell-1} + 6d_{\ell-1}(x_\ell - x_{\ell-1})$$
$$= 2c_\ell$$

exterior point 2

$n-2$ interior points $2 + 2$

polynomial fit

# of equations

$$2 + (n-2)\cdot 4 = 4n - 6$$

# of unknowns:

$(n-1)\cdot 4 = 4n - 4$ unknowns

Natural Spline

$$S_1''(x_1) = S_{n-1}''(x_n) = 0$$

Clamped Spline

$$S_1'(x_1) = y_1'$$

$$S_{n-1}'(x_n) = y_n'$$

Not-a-knot:

$$S_1'''(x_2) = S_2'''(x_2)$$

$$S_{n-1}'''(x_{n-1}) = S_{n-2}'''(x_{n-1})$$

p

$x_1 = 0 \qquad y_1 = 1 \qquad (0, 1)$

$x_2 = 1/2 \, ; \, y_2 = -1 \qquad (1/2, -1)$

$x_3 = 1 \qquad y_2 = 2 \qquad (1, 2)$

Natural spline

$S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$

$S_2(x) = a_2 + b_2 (x - 1/2) + c_2 (x - 1/2)^2 + d_2 (x - 1/2)^3$

$S_1(x_1) = y_1 = \boxed{1 = a_1}$

$S_1(x_2) = S_2(x_2) = y_2$

$\boxed{a_1 + b_1/2 + c_1/4 + d_1/8 = -1}$

$a_2 = y_2 \, ; \, \boxed{a_2 = -1}$

$S_1'(x) = b_1 + 2c_1 x + 3 d_1 x^2$

$S_1''(x) = 2c_1 + 6 d_1 x$

Q

$S_2'(x) = b_2 + 2c_2(x - 1/2) + 3d_2(x - 1/2)^2$

$S_2''(x) = 2c_2 + 6d_2(x - 1/2)$

$S_1'(x_2) = S_2'(x_2)$

$S_1''(x_2) = S_2''(x_2)$

$$\boxed{\begin{aligned} b_1 + c_1 + 3d_1/4 &= b_2 \\ 2c_1 + \tfrac{6}{2}d_1 &= 2c_2 \end{aligned}}$$

$S_2(x_3) = y_3$

$$\boxed{a_2 + b_2/2 + c_2/4 + d_2/8 = 2}$$

$S_1''(x_1) = S_1''(\emptyset) = \boxed{2c_1 = \emptyset}$

$S_2''(x_3) = \boxed{2c_1 + 6d_1 = \emptyset}$

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$$\begin{pmatrix} 1/2 & 1/8 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/4 & 1/8 \\ 1 & 3/4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 3 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ d_1 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$b_1 = -\frac{13}{2} \quad d_1 = 10$

$b_2 = 1 \quad c_2 = 15 \quad d_2 = -10$

$\cancel{S_1(x) = 1 - \frac{13}{2}}$

$S_1(x) = 1 - \frac{13}{2}x + 10x^3$

$S_2(x) = -1 + (x - 1/2) + 15(x - 1/2)^2 - 10(x - 1/2$