power iterations $\qquad\qquad\qquad\qquad\qquad\qquad$ A

$y_0, \quad \underline{y}_{k+1} = \underline{\underline{A}}\, \underline{y}_k$

inverse power iteration

$y_0, \quad \underline{\underline{A}}\, \underline{y}_{k+1} = \underline{y}_k$

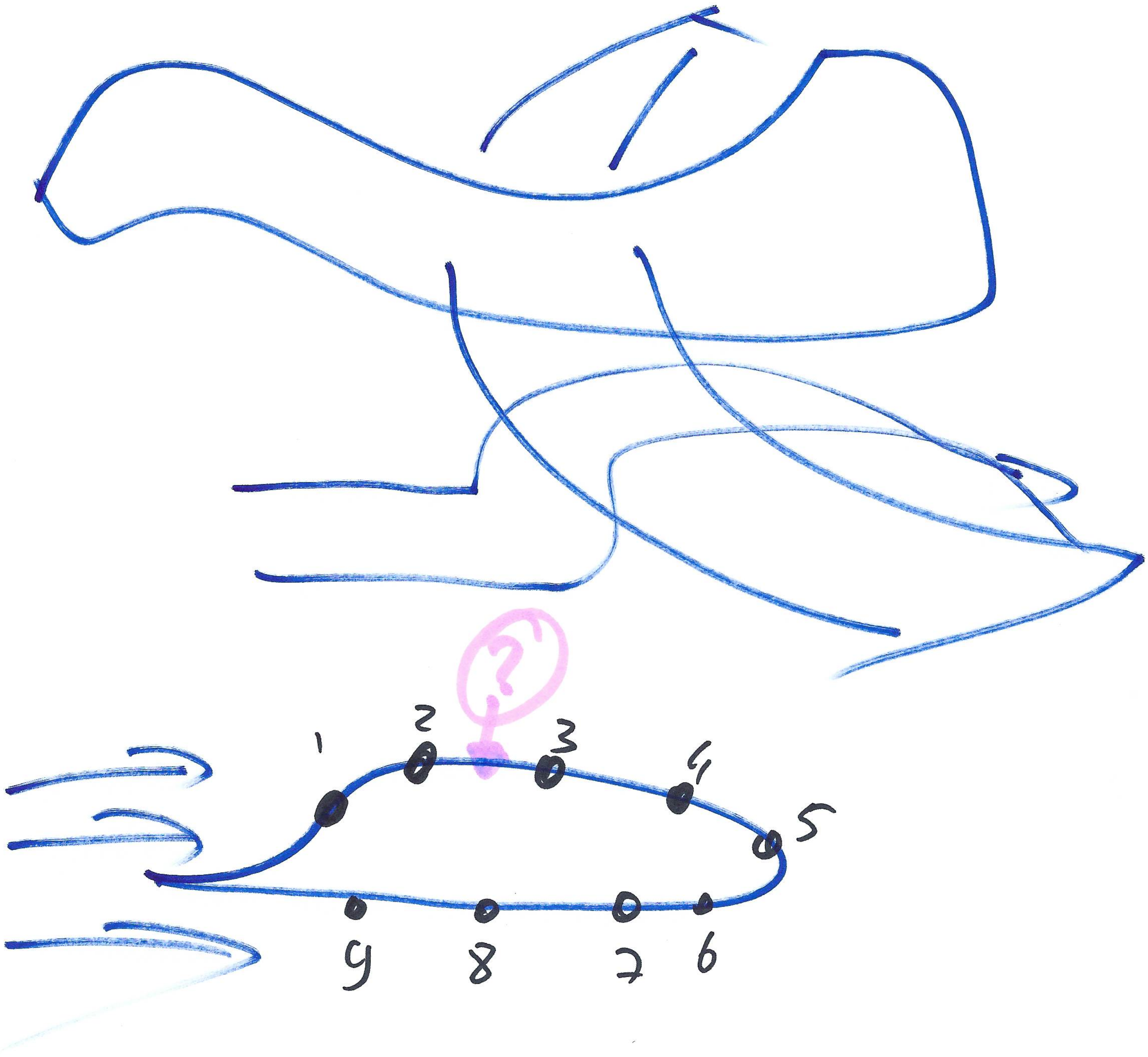
power iterations with a shift.

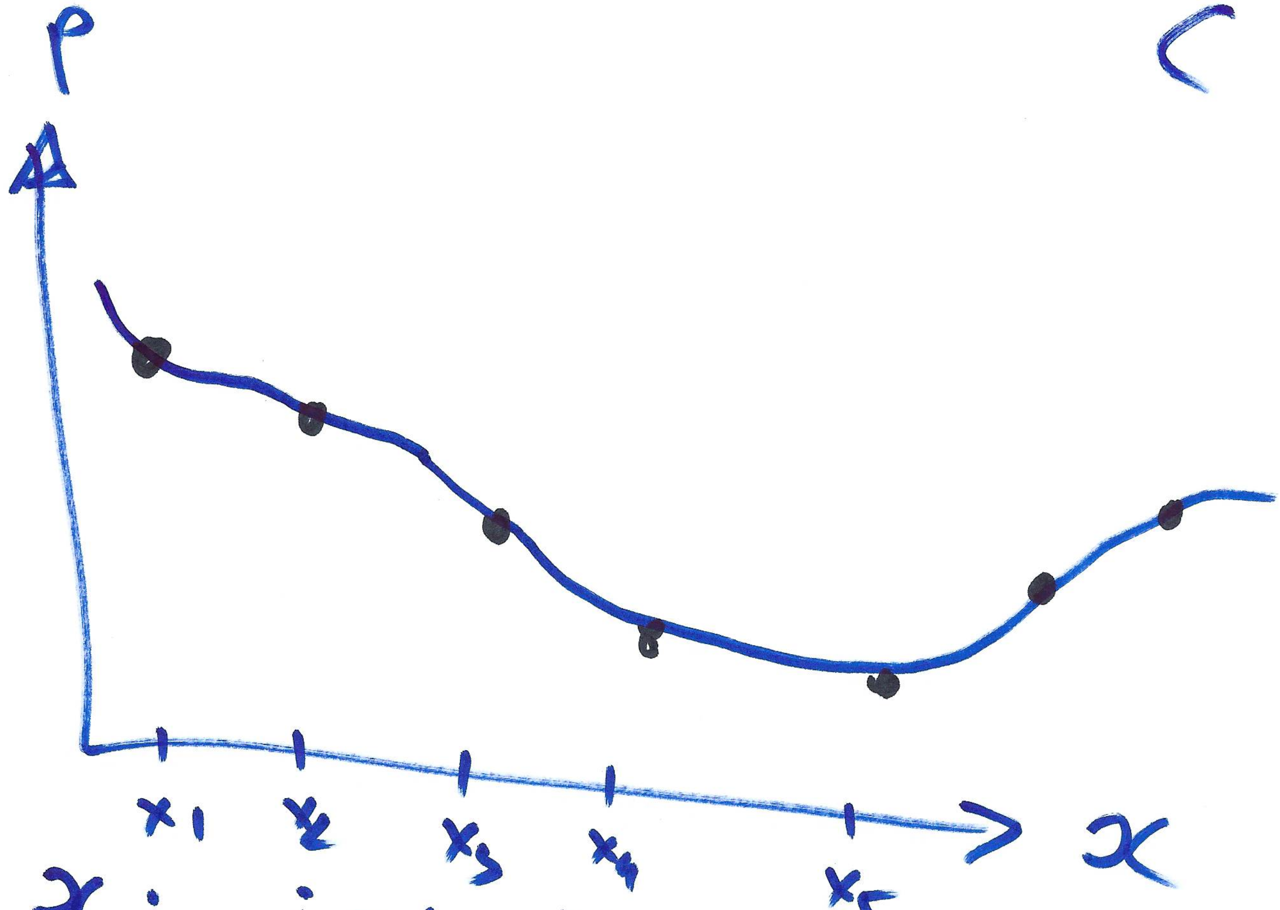
$B = A - I\omega, \quad \omega \in R$

B

Chapter 5

Interpolation





$x_i, \; i = 1..N$

$y_i, \; i = 1..N$

Find function $f(x)$

such that $y_i = f(x_i)$

$i = 1..N$

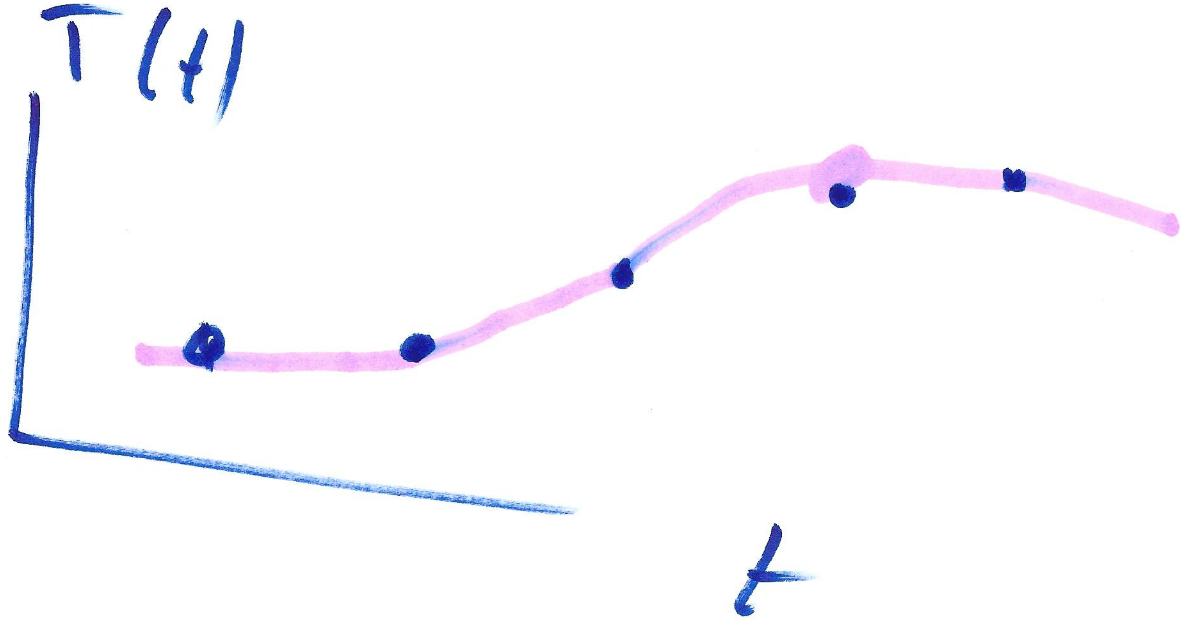
INTERPOLATION

weather.com

D

8 am	43
9 am	47
10 am	49
11am	52

9am 47 mmeters



E

1970 N_1

1980 N_2

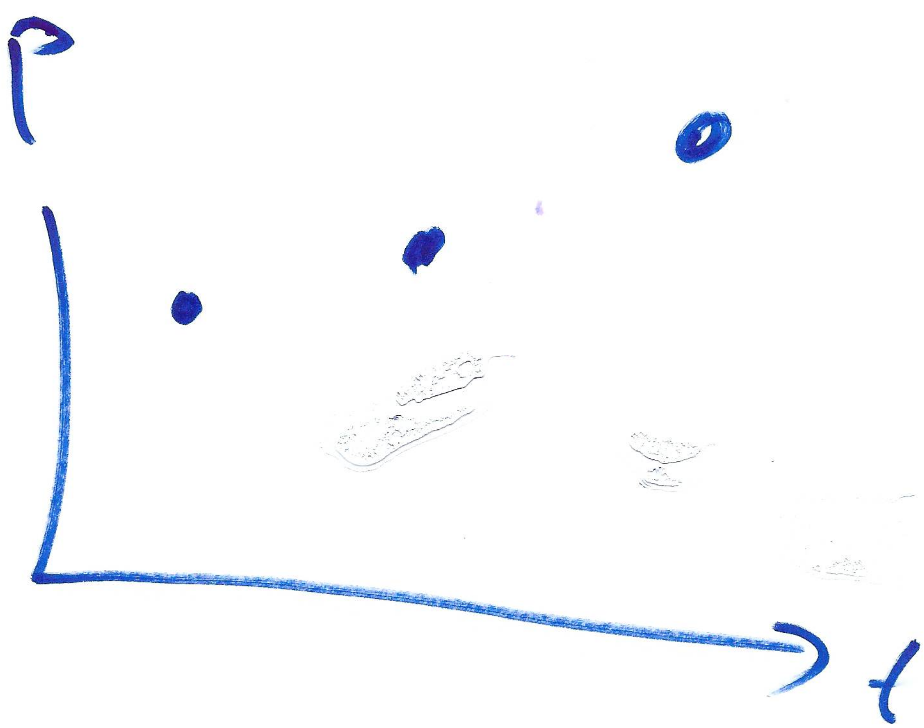
1990 N_3

2000 N_4

2010 N_5

2020 N_6

5 april 1992



"

F

Special Functions

Interpolation:

reading between

the lines

(x_i, y_i) $i = 1..N$

N basis functions

polynomials

trigonometric functions

$P_n(x) / P_m(x)$ P_n - polynomial of degree n

exponential

$(x_i, y_i) \quad i = 1..N$

N basis functions $\varphi_i(x), i = 1..N$

$$f(x) = \sum_{k=1}^{N} a_k \varphi_k(x)$$

$$f(x) = a_1 \varphi_1(x) + a_2 \varphi_2(x) + a_3 \varphi_3(x) + \ldots + a_N \varphi_N(x);$$

Find $a_1, a_2 \ldots a_N$

$$\boxed{\sum_{k=1}^{N} a_k \varphi_k(x_i) = y_i \quad i = 1..N}$$

$$a_1 \varphi_1(x_1) + a_2 \varphi_2(x_1) + a_3 \varphi_3(x_1) + \ldots a_N \varphi_N(x_1) = y_1$$

$$a_1 \varphi_1(x_2) + a_2 \varphi_2(x_2) + a_3 \varphi_3(x_2) + \ldots + a_N \varphi_N(x_2) = y_2$$

$$\cdots$$

$$a_1 \varphi_1(x_k) + a_2 \varphi_2(x_k) + \ldots + a_N \varphi_N(x_k) = y_k$$

$$\cdots$$

$$a_1 \varphi_1(x_N) + a_2 \varphi_2(x_N) + \ldots + a_N \varphi_N(x_N) = y_N$$

$$\begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_N(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_N(x_2) \\ \cdot & — & — & — \\ \varphi_1(x_N) & \varphi_2(x_N) & \dots & \varphi_N(x_N) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \overset{I}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}$$

$$A = (a_{ij}) \; ; \text{ in } \quad a_{ij} = \varphi_j(x_i)$$

$$i,j = 1 .. N$$

$$\underline{a} = (a_i)' \quad i = 1 .. N$$

$$\underline{y} = (y_i)' \, , \quad i = 1 .. N$$

$$\underline{\underline{A}} \; \underline{a} = \underline{y}$$

Monomial interpolation

$$\varphi_m(x) = x^{m-1}$$

$$m = 1 .. N$$

$$(x_i\ y_i) \Rightarrow i = 1 .. N$$

$$f(x) = a_1 \varphi_1(x) + a_2 \varphi_2(x) + .. + a_N \varphi_N(x)$$
$$= a_1 + a_2 x + a_3 x^2 + .. + a_N x^{N-1}$$

$$a_1 + a_2 x_1 + a_3 x_1^2 + .. + a_N x_1^{N-1} = y_1$$
$$a_1 + a_2 x_2 + a_3 x_2^2 + .. + a_N x_2^{N-1} = y_2$$

$$\cdots$$

$$a_1 + a_2 x_N + a_3 x_N^2 + .. + a_N x_N^{N-1} = y_N$$

$$
\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\
1 & x_2 & x_2^2 & & x_2^{N-1} \\
1 & x_3 & x_3^2 & & x_3^{N-1} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & x_N & x_N^2 & & x_N^{N-1}
\end{bmatrix}
\begin{bmatrix}
a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N
\end{bmatrix}
$$

Vandermonde matrix

$$\underline{\underline{V}} = (v_{ij})$$

$$v_{ij} = x_i^{j-1}$$

L

$x_1 = \emptyset$; $y_1 = 1$;

$x_2 = 1$; $y_2 = 2$;

$x_3 = 2$; $y_3 = 5$;

$(\emptyset, 1), (1,2), (2,5)$

$v_{ij} = x_i^{j-1}$; $N = 3$

$f(x) = a_1 + a_2 x + a_3 x^2$

$f(x_1 = \emptyset) = a_1 = 1 = y_1$

$f(x_2 = 1) = a_1 + a_2 \cdot 1 + a_3 \cdot 1^2 =$

$= a_1 + a_2 + a_3 = 2$

$f(x_3 = 2) = a_1 + a_2 \cdot x_3 + a_3 x_3^2 =$

$= a_1 + a_2 \cdot 2 + a_3 \cdot 4 = 5 = y_3$

$\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}; \; \underline{y} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ M

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$a_1 = 1;$

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$2a_2 + 2a_3 = 1 \cdot 2$

$2a_2 + 4a_3 = 4$ $\qquad 2a_3 = 2$

$a_3 = 1$

$a_2 = 0$

$$\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$(-1,0)$ $(0,1)$ $(1,4)$ $(2,15)$

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 15 \end{pmatrix}$$

(x_2, y_2)

0

(x_1, y_1)

Straight line

$$f(x) = ax + b$$

$$f(x_1) = ax_1 + b = y_1$$

$$f(x_2) = ax_2 + b = y_2$$

$$b = y_1 - ax_1$$

$$ax_2 + y_1 - ax_1 = y_2$$

$$a(x_2 - x_1) = y_2 - y_1 \Rightarrow a = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 =$$

$$= \frac{x_2 y_1 - \cancel{y_1 x_1} - x_1 y_2 + \cancel{x_1 y_1}}{x_2 - x_1} = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

$$f(x) = \frac{y_2 - y_1}{x_2 - x_1} \cdot x + \frac{x_2 y_1 - x_1 y_2}{\ }$$

P

$$f(x) = \frac{y_2 - y_1}{x_2 - x_1} \cdot x + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

$$= - \frac{y_1}{x_2 - x_1} \cdot x + y_1 \cdot \frac{x_2}{x_2 - x_1}$$

$$+ \cancel{y_2} \frac{y_2}{x_2 - x_1} X - \frac{x_1 y_2}{x_2 - x_1}$$

$$= y_1 \cdot \frac{x_2 - x}{x_2 - x_1} + y_2 \frac{x_1 - x}{x_1 - x_2}$$

$(x_1, y_1) \quad (x_2 \; y_2)$

$$\ell_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\ell_2(x) = \frac{x_1 - x}{x_1 - x_2}$$

$$\boxed{\begin{pmatrix} \ell_1(x_1) & \ell_2(x_1) \\ \ell_1(x_2) & \ell_2(x_2) \end{pmatrix} = A \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} \ell_1(x_1) & \ell_2(x_1) \\ \ell_2(x_1) & \ell_2(x_2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Q

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$a_1 = y_1 \quad a_2 = y_2$