$\underline{A}$ $\quad \underline{\underline{A}}\, \underline{x} = \lambda \underline{x}$; eigenvalues $\lambda_1 \dots \lambda_n$

$|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

Power method eigenvectors $x_1 \dots x_n$

y_0 - random n vector

$A \in R^{n \times n}$

$\underline{y}_1 = \underline{\underline{A}}\, \underline{y_0}$; $\underline{y}_1 = y_1 / \|y_1\|_2$

$\underline{y_2} = \underline{\underline{A}}\, \underline{y_1}$; $\underline{y_2} = \underline{y_2} / \|y_2\|_2$

$\dots$

$\underline{y_{k+1}} = \underline{\underline{A}}\, \underline{y_k}$; $y_{k+1} = y_{k+1} / \|y_{k+1}\|_2$

$$\underline{y_0} = \sum_{i=1}^{n} \alpha_i \underline{x_i}$$

$$\underline{y_1} = \underline{\underline{A}}\, \underline{y_0} = \sum_{i=1}^{n} \alpha_i \lambda_i \underline{x_i}$$

$$\underline{y_k} = \sum_{i=1}^{n} \alpha_i \lambda_i^{k} \underline{x_i} =$$

$$= \alpha_1 \lambda_1^{k} \underline{x_1} + \sum_{i=2}^{n} \alpha_i \lambda_i^{k} \underline{x_i}$$

$$y_k = \lambda_1^k \left(\alpha_1 \underline{x}_1 + \sum_{i=2}^{n} \alpha_i \left(\frac{\lambda_i}{\lambda_1} \right)^k x_i \right)$$

$$\lim_{k \to \infty} y_k = \lim_{k \to \infty} \lambda_1^k \Big(\alpha_1 \underline{x}_1 + \underbrace{\sum_{i=2}^{n} \alpha_i \left(\frac{\lambda_i}{\lambda_1} \right)^k x_i}_{\to \emptyset} \Big)$$

$$= \lim_{i \to \infty} \lambda_1^k \alpha_1 \underline{x}_1$$

How fast Power method converge?

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \text{ Choose } A = A^T;$$

$a_{12} = a_{21}$

Theorem 4.1: IF $A = A^T$

Then $\lambda_1 > \emptyset$, $\lambda_2 > \emptyset$, and

$$x_i^T x_j = \begin{cases} 1, & i = j \\ \emptyset, & i \neq j \end{cases}$$

$$Ax = \lambda x \Leftarrow x_i^T A x_i = \lambda x_i^T x_i$$

or $\lambda = \dfrac{x_i^T A x_i}{x_i^T x_i}$

$$y_0 = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 ;$$

D

$$v_k = \frac{x_k^T A x_k}{x_k^T x_k}$$ - approximation to the dominant eigenvalue.

$$x_k = \alpha_1 \lambda_1^k \underline{x}_1 + \alpha_2 \lambda_2^k \underline{x}_2$$

$$v_k = \frac{x_k^T A x_k}{x_k^T x_k} =$$

$$\cancel{= \alpha_1 x_1^T \cdot \lambda_1^k A}$$

$$= \frac{(\alpha_1 \lambda_1^k \underline{x}_1^T + \alpha_2 \lambda_2^k \underline{x}_2^T)(A)(\alpha_1 \lambda_1^k x_1 + \alpha_2 \lambda_2^k x_2)}{(\alpha \lambda_1^k \underline{x}_1^T + \alpha_2 \lambda_2^k x_2^T)(\alpha_1 \lambda_1^k x_1 + \alpha_2 \lambda_2^k x_2)}$$

$$\nu_k = \frac{(\alpha_1 \lambda_1^k \underline{x}_1^T + \alpha_2 \lambda_2^k \underline{x}_2^T)(\alpha_1 \lambda_1^{k+1} \underline{x}_1 + \alpha_2 \lambda_2^{k+1} \underline{x}_2)}{(\alpha_1 \lambda_1^k \underline{x}_1^T + \alpha_2 \lambda_2^k \underline{x}_2^T)(\alpha_1 \lambda_1^k \underline{x}_1 + \alpha_2 \lambda_2^k \underline{x}_2)} = \quad \Leftarrow$$

$$= N/D$$

$$N = \alpha_1^2 \lambda_1^{2k+1} \underline{x}_1^T \underline{x}_1 + \alpha_1 \alpha_2 \lambda_1^k \lambda_2^{k+1} \underline{x}_1^T \underline{x}_2 + \alpha_1 \alpha_2 \lambda_1^{k+1} \lambda_2^k \underline{x}_2^T \underline{x}_1 + \alpha_2^2 \lambda_2^{2k+1} \underline{x}_2^T \underline{x}_2$$

$$x_1^T x_1 = 1 \qquad x_2^T x_2 = 1 \qquad x_1^T x_2 = \emptyset \qquad x_2^T x_1 = \emptyset$$

$$= \alpha_1^2 \lambda_1^{2k+1} + \alpha_2^2 \lambda_2^{2k+1}$$

$$D = \alpha^2 \lambda_1^{2k} \underline{x}_1^T \underline{x}_1 + \alpha_2^2 \lambda_2^{2k} \underline{x}_2^T \underline{x}_2 + \alpha_1 \alpha_2 \lambda_1^k \lambda_2^k x_1^T x_2 + \alpha_1 \alpha_2 \lambda_1^k \lambda_2^k \underline{x}_2^T \underline{x}_1$$

$$= \alpha_1^2 \lambda_1^{2k} + \alpha_2^2 \lambda_2^{2k}$$

$$\omega = \frac{\lambda_2}{\lambda_1} \qquad \alpha = \alpha_2/\alpha_1$$

$$\nu_k = \frac{\alpha_1^2 \lambda_1^{2k+1} + \alpha_2^2 \lambda_2^{2k+1}}{\alpha_1^2 \lambda_1^{2k} + \alpha_2^2 \lambda_2^{2k}} = \lambda_1 \frac{1 + \alpha^2 \omega^{2k+1}}{1 + \alpha^2 \omega^{2k}}$$

$$v_k = \left(\frac{\alpha_1^2 \lambda_1^{2k+1} + \alpha_2^2 \lambda_2^{2k+1}}{\alpha_1^2 \lambda_1^{2k} + \alpha_2^2 \lambda_2^{2k}} \right) \frac{\dfrac{1}{\lambda_1^{2k} \alpha_1^2}}{\dfrac{1}{\lambda_1^{2k} \cdot \alpha_1^2}} \qquad T$$

$$= \frac{\alpha_1^2/\alpha_1^2 \, \dfrac{\lambda_1^{2k+1}}{\lambda_1^{2k}} + \dfrac{\alpha_2^2}{\alpha_1^2} \dfrac{\lambda_2^{2k+1}}{\lambda_1^{2k}} \quad \lambda_1 \cdot \dfrac{\lambda_2}{\lambda_1}}{\dfrac{\alpha_1^2 \lambda_1^{2k}}{\alpha_1^2 \lambda_1^{2k}} + \dfrac{\alpha_2^2 \lambda_2^{2k}}{\alpha_1^2 \lambda_1^{2k}}}$$

$$= \frac{\lambda_1 + \alpha^2 \lambda_1 \cdot \omega^{2k+1}}{1 + \alpha^2 \omega^{2k}} =$$

$$= \lambda_1 \frac{1 + \alpha^2 \omega^{2k+1}}{1 + \alpha^2 \omega^{2k}}$$

$$v_k = \lambda_1 \frac{1+\alpha^2\omega^{2k+1}}{1+\alpha^2\omega^{2k}}$$

$$v_k - \lambda_1 = \lambda_1 \frac{1+\alpha^2\omega^{2k+1}}{1+\alpha^2\omega^{2k}} - \lambda_1$$

$$= \lambda_1 \left(\frac{1+\alpha^2\omega^{2k+1}}{1+\alpha^2\omega^{2k}} - 1 \right) =$$

$$= \lambda_1 \frac{1+\alpha^2\omega^{2k+1} - 1 - \alpha^2\omega^{2k}}{1+\alpha^2\omega^{2k}} =$$

$$= \frac{\lambda_1 \alpha^2 \omega^{2k}(\omega - 1)}{1+\alpha^2\omega^{2k}}$$

Large k $\quad 1+\alpha^2\omega^{2k} \approx 1$

$$v_k - \lambda_1 \approx \lambda_1 \alpha^2 \omega^{2k}(\omega - 1)$$

$$v_{k+1} - \lambda_1 \approx \lambda_1 \alpha^2 \omega^{2k+2}(\omega - 1)$$

$$\frac{v_{k+1} - \lambda_1}{v_k - \lambda_1} = \omega^2; \quad \boxed{v_{k+1} - \lambda_1 \simeq \omega^2 (v_k - \lambda_1)}$$

Theorem 4.3

Let A be symmetric nonzero matrix, with dominant eigenvalue λ_1. Then if initial random y_0 has component in x_1 direction, then power method will converge to x_1, λ_1.

Moreover,

$$|v_k - \lambda_1| = \omega_k^2 \, |v_{k-1} - \lambda_1|,$$

$$|v_k - v_{k-1}| = \tilde{\omega}_k^2 \, |v_{k-1} - v_{k-2}|$$

both $\omega_k, \tilde{\omega}_k \xrightarrow[k \to \infty]{} \left|\frac{\lambda_2}{\lambda_1}\right|$

$$y_k = \alpha_1 \lambda_1^k x_1 + \alpha_2 \lambda_2^k x_2 + \dots + \alpha_n \lambda_n^k x_n \quad \text{I}$$

$$= \lambda_1^k \left(\alpha_1 x_1 + \alpha_2 \left(\lambda_2/\lambda_1\right)^k x_2 + \dots + \alpha_n \left(\lambda_n/\lambda_1\right)^k x_n \right)$$

Let $\alpha_1 = \emptyset$

Theorem 4.4

Let A is $\mathbb{R}^{n\times n}$ with eigenvalues

$\lambda_1, \lambda_2, \dots \lambda_n$ and eigenvectors $x_1, \dots x_n$

① Let ω be a real #,
compose $B = A - I\omega$; I is unity matrix
Then B will have eigenvalues
$\lambda_1 - \omega, \lambda_2 - \omega, \dots \lambda_n - \omega$ with
eigenvectors $x_1 \dots x_n$.

② Let A be invertable (A^{-1} exists $\exists$)
Then eigenvalues of
A^{-1} will be $1/\lambda_1, 1/\lambda_2, \dots 1/\lambda_n$
and eigenvectors will be
$x_1 \dots x_n$

K

$\underline{\underline{A}}\,\underline{x}_i = \lambda\,\underline{x}_i$ 1

$\omega(\underline{\underline{I}}\,\underline{x}_i = \underline{x}_i)$ 2 I is unity matrix

$\underline{\underline{I}}\,\omega\,\underline{x}_i = \omega\,\underline{x}_i$ 3

$A x_i = \lambda_i x_i$ 4

$-$

$I \omega x_i = \omega x_i$ 5

$(A - I\omega) x_i = (\lambda_i - \omega) x_i$ 6

$\underline{\underline{A}}^{-1}(\underline{\underline{A}}\,\underline{x}_i = \lambda\,\underline{x}_i)$

$A^{-1} A x_i = x_i = A^{-1} \lambda_i x_i$

$A^{-1} x_i = \frac{1}{\lambda_i} x_i$

Power iterations

Inverse Power iteration:

Choose y_0

Solve $\underline{\underline{A}}\, y_1 = y_0$

Solve $\underline{\underline{A}}\, y_2 = y_1$

$\cdots$

Solve $\underline{\underline{A}}\, y_{k+1} = y_k$

A - closest to ω

$A - I\omega$