A 4PA

Norms of vectors

Norms of matrices

$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

$\|x\|_2 = 1$

$\xrightarrow{A}$

Ax

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$$\kappa(I) = 1$$

$\kappa($A is singular$) = \infty$

B

$$Ax = b$$

$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\|$$

$$\boxed{\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}}$$

$$A(x + \Delta x) = b + \Delta b$$

$$\underline{\underline{Ax}} + A\Delta x = \underline{\underline{b}} + \Delta b$$

$$A\Delta x = \Delta b$$

$$\cancel{\|\Delta b\| = \|A\Delta x\| \leq \|A\| \cdot \|\Delta x\|}$$

$$\Delta x = A^{-1}\Delta b$$

$$\boxed{\|\Delta x\| = \|A^{-1}\Delta b\| \leq \|A^{-1}\| \cdot \|\Delta b\|}$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}$$

C

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathcal{K}(A)\, \epsilon_{mach}$$

$$Ax = b \Rightarrow A\hat{x} = b$$

$\hookrightarrow$ numerical solution

$$r = b - A\hat{x}$$

$$r = \emptyset \Rightarrow x = \hat{x}$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathcal{K}(A) \frac{\|r\|}{\|b\|}$$

$$\hat{x} = x + \Delta x$$

$$r = b - A(x + \Delta x) = b - Ax - A\Delta x$$

$$= -A\Delta x$$

$$\Delta x = -A^{-1} r$$

$$\|\Delta x\| = \|A^{-1} r\| \leq \|A^{-1}\| \cdot \|r\|.$$

$$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

D

$|A| \leq |B| \qquad 1 \leq 2$

$|C| \leq |D| \qquad 2 \leq 4$

$1 \leq 8$

$|A| \cdot |C| \leq |B| \cdot |D|$

E

$1 \leq x \leq \infty$

Nonlinear system of equations

$$\left.\begin{aligned} f_1(x_1\, x_2 \ldots x_n) &= 0 \\ f_2(x_1\, x_2 \ldots x_n) &= 0 \\ &\ldots \\ f_n(x_1 \ldots x_n) &= 0 \end{aligned}\right\}$$

n equations, n unknowns

Assume solution exists

$\underline{x} = (x_1 \ldots x_n)$

$$\underline{f}(\underline{x}) = \begin{pmatrix} f_1(x_1 \ldots x_n) \\ f_2(x_1 \ldots x_n) \\ f_n(x_1 \ldots x_n) \end{pmatrix}$$

$\underline{f}(\underline{x}) = 0$

"Simplistic" method G

~~1st eg~~

$$\underline{X}^{(0)} = (x_1^{(0)}\ x_2^{(0)} \ldots x_n^{(0)})$$

1st **eg**

$$x_1^{(k+1)} = x_1^{(k)} - \frac{f_1(x_1^{(k)} \ldots x_n^{(k)})}{\frac{\partial f_1}{\partial x_1}(x_1^{(k)} \ldots x_n^{(k)})}$$

$$x_2^{(k+1)} = x_2^{(k)} - \frac{f_2(x_1^{(k)} \ldots x_n^{(k)})}{\frac{\partial f_2}{\partial x_2}(x_1^{(k)} \ldots x_n^{(k)})}$$

$$x_i^{(k+1)} = x_i^{(k)} - \frac{f_k(x_1^{(k)} \ldots x_n^{(k)})}{\frac{\partial f_i}{\partial x_i}(x_1^{(k)} \ldots x_n^{(k)})}$$

$\underline{f}(\underline{x}) = \emptyset$

H

$i = 1 \ldots n$

Guess $\underline{x}^{(0)}$

iterations

$$x_i^{(k+1)} = x_i^{(k)} - \frac{f_i(x_1^{(k)} \ldots x_n^{(k)})}{\left(\dfrac{\partial f_i(x_1^{(k)} \ldots x_n^{(k)})}{\partial x_i} \right)}$$

I

$$\left.\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right\}$$

$$(x^{(0)}, y^{(0)}) \ldots (x^{(k)}, y^{(k)})$$

$\cancel{g}($

$$\emptyset = f(x_{k+1}, y_{k+1}) = f(x_k + x_{k+1} - x_k, y_k + y_{k+1} - y_k) =$$

$$\approx f(x_k, y_k) + \frac{\partial f(x_k, y_k)}{\partial x}(x_{k+1} - x_k) + \frac{\partial f(x_k, y_k)}{\partial y}(y_{k+1} - y_k)$$

$$\emptyset = g(x_{k+1}, y_{k+1}) = g(x_k + x_{k+1} - x_k, y_k + y_{k+1} - y_k) =$$

$$\approx g(x_k, y_k) + \frac{\partial g(x_k, y_k)}{\partial x}(x_{k+1} - x_k) + \frac{\partial g(x_k, y_k)}{\partial y}(y_{k+1} - y_k)$$

$$\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}_{(x_k, y_k)} \begin{pmatrix} x_{k+1} - x_k \\ y_{k+1} - y_k \end{pmatrix} = - \begin{pmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{pmatrix}$$

$$x_{k+1} - x_k = \Delta x_{k+1}$$

$$y_{k+1} - y_k = \Delta y_{k+1}$$

$$\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} \begin{pmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = - \begin{pmatrix} f(x_k\ y_k) \\ g(x_k\ y_k) \end{pmatrix}^{k}$$

Algorithm

Choose (x_0, y_0)

Calculate Jacobian

$$\begin{pmatrix} \partial f(x,y)/\partial x & \partial f(x,y)/\partial y \\ \partial g(x,y)/\partial x & \partial g(x,y)/\partial y \end{pmatrix} = \underline{\underline{J}}(x,y)$$

While $\Delta x^2 + \Delta y^2 \leq (\text{TOLERANCE})^\wedge 2$

$\underline{\underline{J}} = J(x_k, y_k)$

Solve $\underline{\underline{J}} \cdot \begin{pmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = - \begin{pmatrix} f(x_k\ y_k) \\ g(x_k\ y_k) \end{pmatrix}$

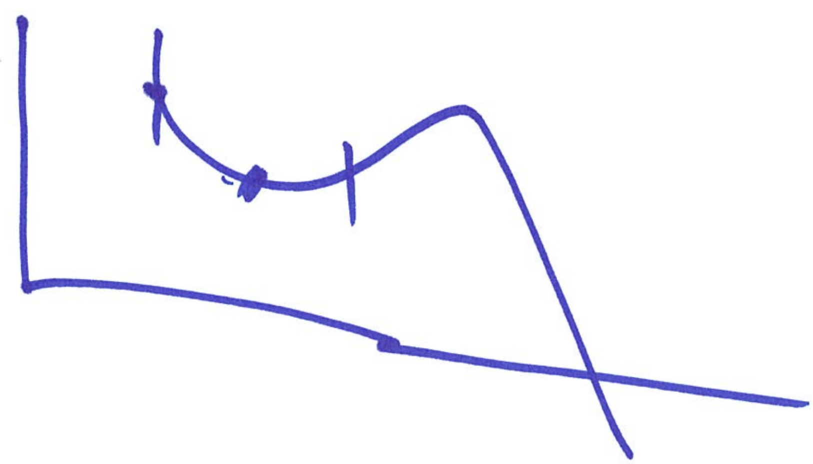
$x_{k+1} = x_k + \Delta x_{k+1}$

$y_{k+1} = y_k + \Delta y_{k+1}$ ENDWHILE

L

$$\sum_{i=1}^{n} \left(\Delta x_i^{(k)} \right)^2 \leq \text{TOLERANCE}^2$$

- $\| \underline{f}(\underline{x}) \| = \left\| \begin{pmatrix} f_1(x_1 \ldots x_n) \\ \vdots \\ f_n(x_1 \ldots x_n) \end{pmatrix} \right\| \leq \text{TOLER}$
- Max # of iterations reached



Largest integer n to represent

M

$$2^n + 2^{-n}$$

10

$\textcircled{2}$

$n = 0, \quad 2^0 + 2^0 = 2 = 10$

$n = 1 \quad 2^1 + 1/2 = \quad 10.1_{[2]}$ 3 digits

$n = 2 \quad 4 + 1/4 = 100.01_{[2]}$ 5 digits

$n = 3 \quad 8 + 1/8 = 1000.001$ 7 digits

$n = 4 \quad 16 + 1/16 \quad 10000.0001$ 9 digits

$\#$ digits $= 2n + 1 = 23 < 24$

$n = 11$

N

$$(1+(2^{-24}+2^{-25}))^4-1$$

$$=1+2^{-24}(1+1/2)$$

$$N=24,\ \epsilon=2^{-23}$$

$$1.\underbrace{000}_{23}11$$

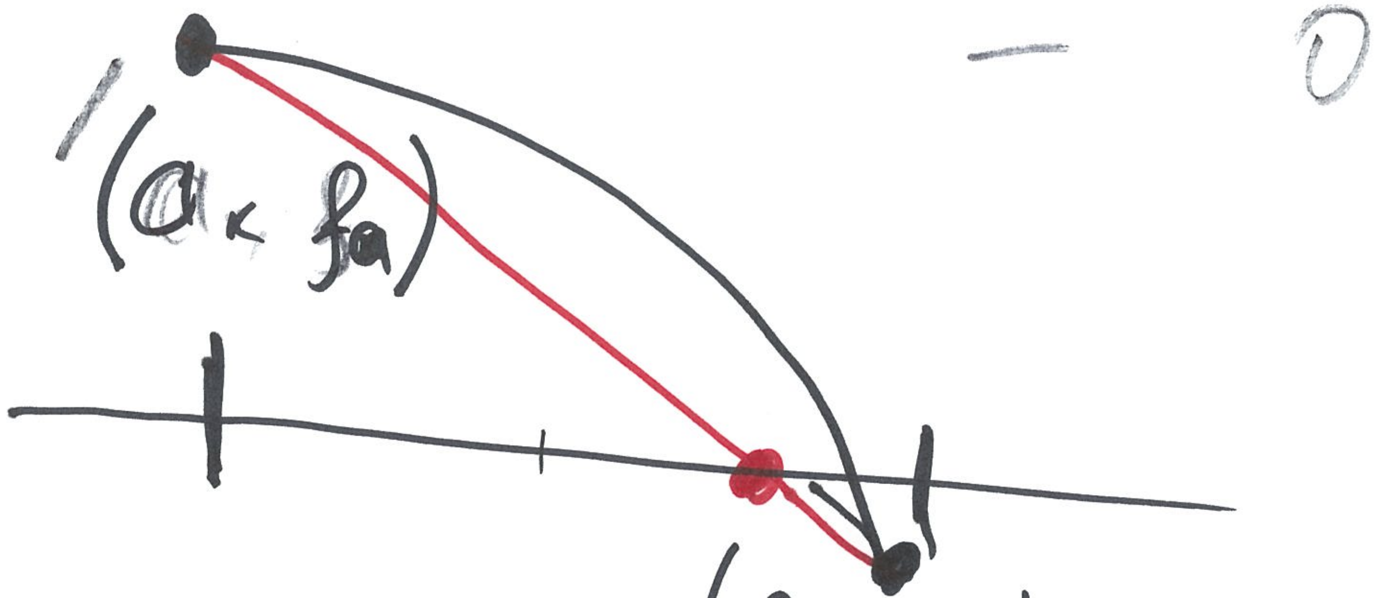
$$(1.\underbrace{000}_{22}1)^4=1.\underbrace{\quad}_{20}1$$

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b)^4=(a^2+2ab+b^2)^2=$$

$$=a^4+b^4+4a^3b+4ab^3+2a^2b^2$$

$$a=1,\ b=2^{-23}$$



$$y = \alpha x + \beta = 0 \qquad x = -\beta/\alpha$$

$$\left.\begin{aligned} f_a &= \alpha a_k + \beta \\ f_b &= \alpha b_k + \beta \end{aligned}\right\} \alpha, \beta$$

$$f_a - \alpha a_k = f_b - \alpha b_k$$

$$\alpha (a_k - b_k) = f_a - f_b$$

$$\alpha = \frac{f_a - f_b}{a_k - b_k}$$

$$\beta = f_a - \alpha a_k = \frac{\cancel{f_a a_k} - f_a b_k - \cancel{a_k f_a} + a_k f_b}{a_k - b_k}$$

$$x_{k+1} = -\frac{\beta}{\alpha} = \frac{f_a b_k - a_k f_b \cancel{(a_k - b_k)}}{\cancel{(a_k - b_k)}(f_a - f_b)}$$

n equations, n unknowns P

$$\underline{f}(\underline{x}) = \underline{0}$$

$$\underline{f}(\underline{x}) = \begin{pmatrix} f_1(x_1 \ldots x_n) \\ \vdots \\ f_n(x_1 \ldots x_n) \end{pmatrix}$$

$\underline{x}^{(0)}$, after k iterations

$$\underline{x}^{(k)}$$

$$\cancel{f(x_{k+}} \quad \underline{f}(\underline{x}^{k+1}) =$$

$$= \underline{f}(\underline{x}_k + \underline{x}_{k+1} - \underline{x}_k)$$

$$= \underline{f}(\underline{x}_k) + \underline{\underline{J}} \cdot (\underline{x}_{k+1} - \underline{x}_k) = \underline{0}$$

$$\underline{\underline{J}} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \ldots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \ldots & \ldots & \partial f_2/\partial x_n \\ \partial f_n/\partial x_1 & & & \partial f_n/\partial x_n \end{pmatrix}$$

$$\underline{\underline{J}}(\underline{x}_k) \cdot \Delta \underline{x}_{k+1} = -\underline{f}(\underline{x}_k)$$

$$\underline{\Delta x_{k+1}} = \underline{x}_{k+1} - \underline{x}_k$$

$$J^{-1}\left[\underline{\underline{J}}(\underline{x}_{k+1} - \underline{x}_k) = -\underline{f}(\underline{x}_k)\right]$$

$$\underline{x}_{k+1} = \underline{x}_k - \underline{\underline{J}}^{-1}(\underline{x}_k)\,\underline{f}(\underline{x}_k)$$

T1 - 18

T2 3

T3 6

T4 19