A

LU decomposition

$$A \to \underline{M_1} \cdot \quad a_{11}$$

$$M_2 \quad \tilde{a}_{22}$$

$$\vdots$$

$$M_1 = \begin{bmatrix} \ddots & & \\ -\frac{a_{21}}{a_{11}} & 1 & \\ -\frac{a_{31}}{a_{11}} & & \ddots \\ -\frac{a_{41}}{a_{11}} & & \\ \vdots & & \end{bmatrix}$$

$$a_{11} = 0$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

B

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} x + 2y + 3z = 10 \\ 2x + 3y + 2z = 25 \\ 3x + 5y + 7z = 7 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 25 \\ 7 \end{pmatrix}$$

$\Rightarrow$ changing rows does NOT affect a solution

partial pivoting is a process of changing order of rows to have a_{ii} be biggest possible.

C

Theorem 3.1

Any n x n matrix can be pivoted, so that LU decomposition of matrix exists.

D

Norm of a vector

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\|x\|_p = \left[\sum_{k=1}^{n} |x_k|^p \right]^{1/p}$$

$$\|A\| = \max_{\underline{x} \neq 0} \frac{\|A \cdot x\|}{\|x\|}$$

IF $\|x\| = 1$, $\|A\| = \max_{x \neq 0} \|Ax\|$

Condition number of a matrix.

$$\varkappa(A) = \|A\| \cdot \|A^{-1}\|$$

cond(A)

cond(A, p)   norm

- $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{cond}_1(A) = 1$

- $A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}; \ A^{-1} = \begin{pmatrix} 1/d_1, & 0 \\ 0 & 1/d_2 \end{pmatrix}$

$\|A\|_\infty = \max(|d_1|, |d_2|)$

$\|A^{-1}\|_\infty = \max(|1/d_1|, |1/d_2|)$

F

$$\|A\|_\infty = \max(|d_1|, |d_2|)$$

$$\|A^{-1}\|_\infty = \max(1/|d_1|, 1/|d_2|)$$

$$= \frac{1}{\min(|d_1|, |d_2|)}$$

$$\kappa(A) = \frac{\max(|d_1|, |d_2|)}{\min(|d_1|, |d_2|)}$$

$|d_1|$ is small

$$0 < d_1 \ll 1$$

$d_2 \simeq 1$

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

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- $$A = \begin{pmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ 0 & & & & d_n \end{pmatrix}$$

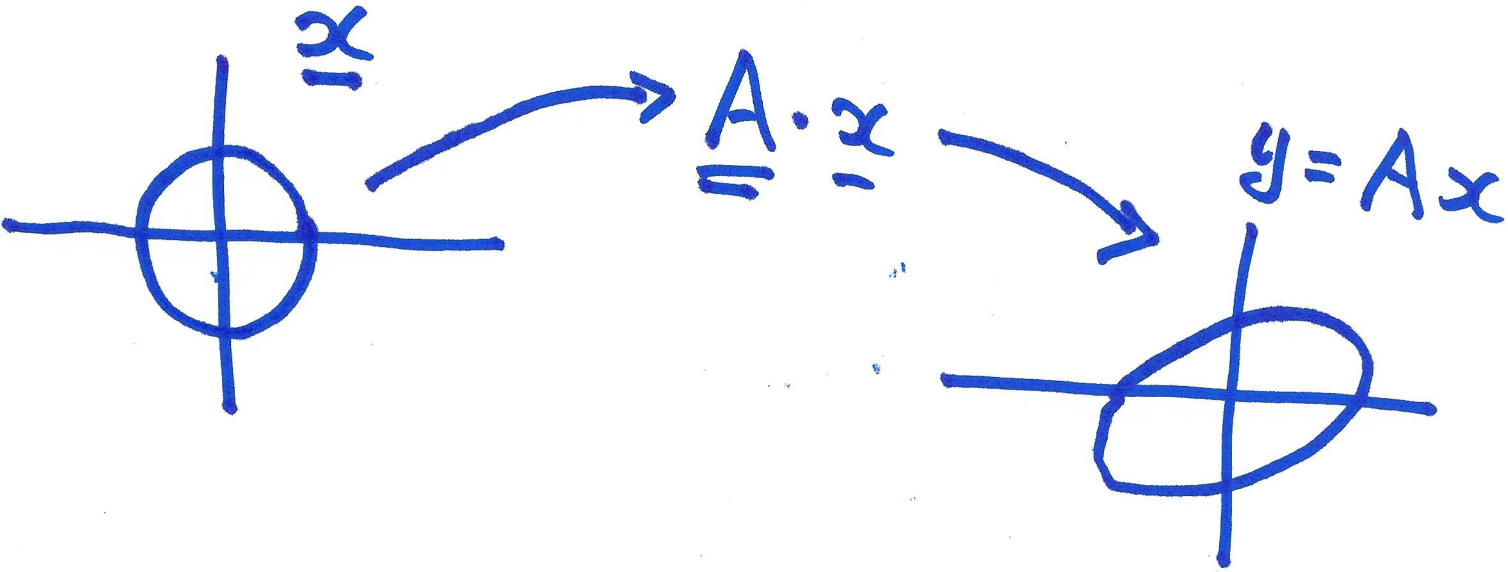
$$A^{-1} = \begin{pmatrix} 1/d_1 & & & 0 \\ & 1/d_2 & & \\ & & \ddots & \\ 0 & & & 1/d_n \end{pmatrix}$$

$$\varkappa(A) = \frac{\max\limits_{i=1..n} |d_i|}{\min\limits_{i=1..n} |d_i|}$$

H

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\|x\|_2 = 1$$



$$z = \underline{\underline{A}}^{-1} \cdot \underline{y} = \underline{\underline{A}}^{-1} \underline{\underline{A}} \cdot \underline{x}$$
$$= x$$

$\cancel{\varkappa}_2$

$$\varkappa_2(A) = \frac{\max |y|}{\min |y|}$$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} a_{ijk}$$ - max column sum

$$\|A\|_\infty = \max_j \sum_{i=1}^{n} a_{ij}$$ - max rowsum

① $\forall A, \quad \varkappa(A) \geq 1$

② IF $A = I$, $\varkappa(A) = 1$

③ IF A is a matrix, γ is a scalar

$$\varkappa(\gamma A) = \varkappa(A)$$

④ IF A is singular ($\det(A) = \emptyset$, or A^{-1} does not exist)

then $\varkappa(A) = \infty$

⑤ IF $A = d_i \delta_j^i$ - diagonal $\varkappa = \frac{\max|d_i|}{}$

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solve $Ax = b$,

solve $A\hat{x} = b + \Delta b$

$\hat{x} = x + \Delta x$

$\Delta x = \hat{x} - x$

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{\overbrace{\|A^{-1}\|\cdot\|A\|}^{\varkappa}\,\|b\|}{\|b\|}$$

$A\hat{x} = A\cdot(x + \Delta x) =$

$= \underline{A\cdot x} + A\cdot\Delta x = \underline{b} + \Delta b$

$A\cdot\Delta x = \Delta b$, $\Delta x = A^{-1}\cdot\Delta b$

$$\boxed{\|\Delta x\| = \|A^{-1}\Delta b\| \le \|A^{-1}\|\cdot\|\Delta b\|}$$

$\|b\| = \|A\cdot x\| \le \|A\|\cdot\|x\|$

$$\boxed{\frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}}$$

$$\frac{\|\Delta x\|}{\|x\|} \le \varkappa(A)\frac{\|\Delta b\|}{\|b\|}$$

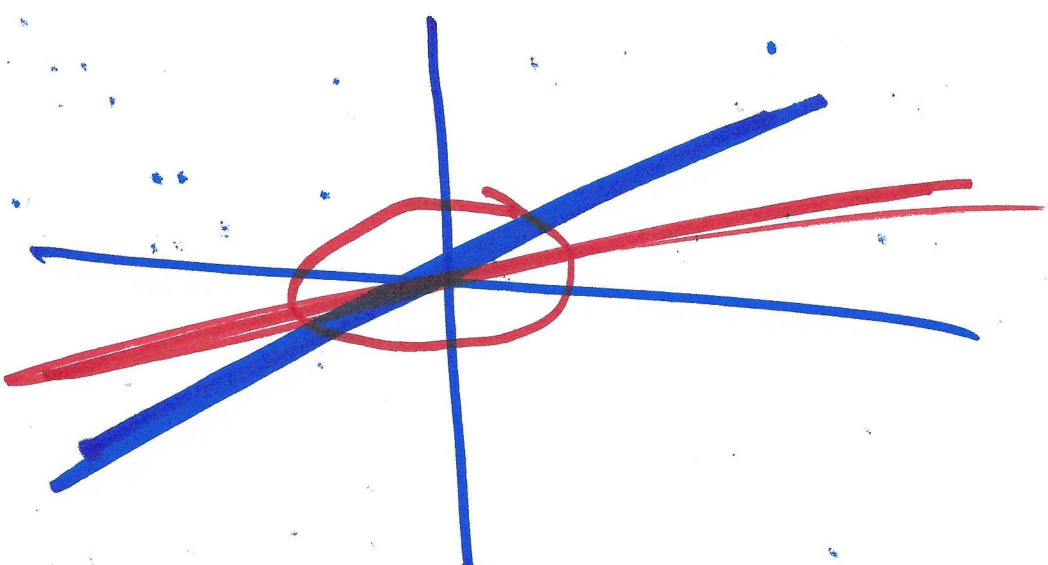
κ

$$Ax = b$$

$$(A + \Delta A)\hat{x} = b$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \le \kappa(A) \frac{\|\Delta A\|}{\|A\|}$$

$$\frac{\|\Delta x\|}{\|x\|} \le \kappa(A)\, \epsilon_{mach}$$



Residual $$\frac{\|\Delta x\|}{\|x\|} \le \mathcal{X}(A)\frac{\|r\|}{\|b\|}$$

Solve $Ax = b$

computer gives $\hat{x}$

$\Delta x = -\hat{x} + x$ - error

$r = b - A\hat{x}$; $A\hat{x} = b - r$

↳ residual $\hat{x} = A^{-1}b - A^{-1}r$

$\Delta x = x - \hat{x} = A^{-1}b - A^{-1}b + A^{-1}r$

$\Delta x = A^{-1}r \qquad = A^{-1}r$

$$\|\Delta x\| = \|A^{-1} \cdot r\| \le \|A^{-1}\| \cdot \|r\|$$

$$\frac{1}{\|x\|} \le \|A\| / \|b\|$$

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{\|A^{-1}\| \cdot \|A\| \|r\|}{\|b\|}$$

Solving nonlinear equations.

$f(x,y) = \emptyset$

$g(x;y) = \emptyset$

Simplistic way

x_0, y_0

1st equation

$$x_1 = x_0 - \frac{f(x_0, y_0)}{\partial f(x_0, y_0)/\partial x}$$

$$y_1 = y_0 - \frac{g(x_0, y_0)}{\partial g(x_0 y_0)/\partial y}$$

N

$$x_{k+1} = x_k - \frac{f(x_k\ y_k)}{\partial f(x_k, y_k)/\partial x}$$

$$y_{k+1} = y_k - \frac{g(x_k\ y_k)}{\partial g(x_k\ y_k)/\partial y}$$