-1-

Calculus of variations

- $f(x) \rightarrow$

$$f'(x) = \frac{df(x)}{dx} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

- $f(x, y)$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y) - f(x,y)}{\varepsilon}$$

- Calculus of variation:

Functional

function of a fuction

$f(x) \longrightarrow$ number

$$I[f(x)] = \int_{-\infty}^{\infty} f(x)\,dx$$

$$I[f(x)] = \int_a^b f(x)\,dx$$

$$I[f(x)] = \int_a^b f(x)\,\overset{\text{given}}{g(x)}\,dx$$

$$I[f(x)] = \int_a^b \frac{d f(x)}{dx}\, g(x)\,dx$$

$$I[f(x)] = \int_a^b \underset{\text{given}}{F}\left(f(x), f'(x), f''(x) \ldots, x \right) dx$$

Variation

$$\delta I[f(x)] = I[f(x) + \varepsilon \eta(x)] - I[f(x)]$$

Boundary condition

conditions for η

$$\eta(a) = \eta(b) = \emptyset$$

$$\frac{\delta I}{\delta \eta} = \frac{I[f + \varepsilon \eta] - I[f]}{\varepsilon \eta}$$

$$\frac{\delta I[f(x)]}{\delta f(x)} \Rightarrow$$

② $\delta I[f(x)] \equiv I[f(x) + \eta(x)] - I[f(x)]$

③ $$= \int_0^1 \frac{\delta I[f(x)]}{\delta f(x)} \eta(x)\, dx$$

$\boxed{\text{EXAMPLE}}$

-5' • $I[f(x)] = \int_a^b f(x) g(x) dx$

$\frac{\delta I[f(x)]}{\delta f(x)} = ?$ $\quad \eta(a) = \eta(b) = \emptyset$

$\delta I = I[f(x) + \eta(x)] - I[f(x)]$

$= \int_a^b (f(x) + \eta(x)) g(x) dx$

$- \int_a^b f(x) g(x) dx$ $\qquad \frac{\delta I[f(x)]}{\delta f(x)} = g(x)$

$= \int_a^b \eta(x) g(x) dx$

$= \int_a^b \frac{\delta I[f(x)]}{\delta f(x)} \eta(x) dx$

$\frac{\delta I[f(x)]}{\delta f(x)}$

-6-

$$\frac{\delta I[f(x)]}{\delta f(x)} \overset{???}{=}$$

$$= \frac{I[f(x)+\eta(x)] - I[f(x)]}{\eta(x)} =$$

$$= \int_a^b \eta(x) g(x)\, dx \Big/ \eta(x)$$

$\eta(x) = \delta(x-a)$

$$I[f(x)] = \int_a^b \frac{df(x)}{dx} g(x)\,dx$$

$$\delta I[f(x)] =$$

$$= \int_a^b dx \left[\left(\frac{df(x)}{dx} + \frac{d\eta(x)}{dx} \right) g(x) - g(x)\frac{df(x)}{dx} \right] =$$

$$= \int_a^b dx\, \frac{d\eta(x)}{dx} g(x) =$$

$$= \int_a^b \overbrace{g(x)}^{u} \underbrace{\frac{d\eta(x)}{dx} \cdot dx}_{dV = d\eta} =$$

$$= \underbrace{g(x)\,\eta(x)\Big|_{x=a}^{x=b}}_{=\emptyset} - \int_a^b \eta(x) \frac{dg(x)}{dx}\, dx$$

8

$$\delta I[f(x)] = -\int_a^b \eta(x)\,\frac{dg(x)}{dx}\,dx$$

$$\stackrel{df}{\equiv} \int_a^b \frac{\delta I}{\delta f(x)}\,\eta(x)\,dx$$

$$\frac{\delta}{\delta f(x)}\int_a^b f'(x)g(x)\,dx$$

$$= -g'(x)$$

- $$\frac{\delta}{\delta f(x)}\int_a^b f''(x)g(x)\,dx =$$

$$I[f(x)] = \int_a^b f''(x)\, g(x)\, dx$$

$$\delta I[f(x)] = \int_a^b dx \Big[f''(x) g(x) + \eta''(x) g(x) - f''(x) g(x) \Big] =$$

$$= \int_a^b dx\; \eta''(x) g(x) =$$

$$= \int dx\; g(x) \frac{d}{dx} \eta'(x)$$

$$= g(x)\, \eta'(x) \Big|_{x=a}^{x=b} - \int_a^b dx\; \eta'(x)\, g'(x)$$

$$= - \int_a^b \overset{u}{g'(x)} \underbrace{\frac{d}{dx} \underbrace{\eta'(x)\, dx}_{V}}_{dV}$$

$$I[f(x)] = \int_a^b F\left(f(x), f'(x), f''(x) \ldots, x\right) dx$$

$$\rightarrow \frac{\partial}{\partial f(x_n)} \sum_{n=1}^{N} F\left(f(x_n), f'(x_n) \ldots f''(x_n), x_n\right)$$

$\rightarrow$ Remove Σ

All mechanical systems

have $\mathcal{L}(\underline{x}(t), \dot{\underline{x}}(t), t)$

for 1 particle

n particle

$$\mathcal{L}(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n, \dot{\underline{x}}_1, \dot{\underline{x}}_2 \ldots \dot{\underline{x}}_n, t)$$

Fluids: $\mathcal{L}(\underline{x}(\bar{r}), \dot{\underline{x}}(\bar{r}), t)$

Action:

$$S[\underline{x}(t)] = \int_{t=t_0}^{t=T} d\tau\, \mathcal{L}(\underline{x}(t), \underline{\dot{x}}(t), t)$$

ALL MECHANICAL SYSTEMS MOVE TO MINIMIZE ACTION

$$\frac{\delta S[x(t)]}{\delta x(t)} = \emptyset$$

$$\delta S(x(t)) = \int_{t=0}^{T} d\tau \left[\mathcal{L}\left(x(\tau)+\eta(\tau),\ \dot{x}(\tau)+\dot{\eta}(\tau),\ \tau\right) \right] - \int_{t=0}^{T} d\tau \left[\mathcal{L}\left(x(\tau), \dot{x}(\tau), \tau\right) \right]$$

$$= \int_{t=0}^{T} d\tau \left[\mathcal{L}(x(\tau), \dot{x}(\tau), \tau) + \frac{\partial \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\partial x(\tau)} \cdot \eta(\tau) + \frac{\partial \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\partial \dot{x}(\tau)} \dot{\eta}(\tau) - \mathcal{L}(x(\tau), \dot{x}(\tau), \tau) \right] =$$

$$= \cancel{\int_{t=0}^{T} d\tau \Big[\partial \mathcal{L}(}$$

$$\delta S = \int_{t=0}^{T} d\tau \left[\frac{\partial \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\partial x(\tau)} \eta(\tau) + \underbrace{\frac{\partial \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\partial \dot{x}} \dot{\eta}(\tau)}_{} \right]$$

$$= \int_{t=0}^{T} d\tau \left[\frac{\partial \mathcal{L}}{\partial x(\tau)} \eta(\tau) - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}} \eta(\tau) \right] + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}} \cdot \eta(\tau) \Big|_{t=0}^{t=T}}_{=0}$$

AT $\tau, x(\tau), \dot{x}(\tau$

$$\equiv \int_{t=0}^{T} d\tau \, \frac{\delta \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\delta x(\tau)} \eta(\tau)\, d\tau$$

$$\frac{\delta S}{\delta x(\tau)} = \frac{\partial \mathcal{L}(x(\tau)\dot{x}(\tau)\tau)}{\partial x(\tau)}$$

$$- \frac{d}{dt} \frac{\partial \mathcal{L}(x(\tau), \dot{x}(\tau), \tau)}{\partial \dot{x}}$$

$$= \emptyset$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \emptyset$$

$$L(x) = \frac{m\dot{x}^2}{2} - V(x) \quad (1)$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V(x)}{\partial x} \quad (2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad 3$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x}$$