Euler - Lagrange

variational calculus.

$$I[f(x)] = \int_a^b F(x, f(x), f'(x))\,dx$$

Function of a function is a functional

$$\frac{\partial F}{\partial f} + \frac{d}{dx}\frac{\partial F}{\partial F'} = 0$$

Euler - Lagrange equation

Let us a functional, such that

$$I[f(x)] = \int_a^b F(f(x), f'(x))\,dx$$

Beltrami Identity

$$B = F\left(f(x), f'(x)\right) - f'(x)\,\frac{\partial F\left(f(x), f'(x)\right)}{\partial f'(x)}$$

$$B = F - f'\,\frac{\partial F}{\partial f'}$$

$$\frac{dB}{dx} = \frac{dF}{dx} - f''(x)\,\frac{\partial F}{\partial f'} - f'\,\frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right)$$

$f(x, y(x))$: $$\frac{d f(x, y(x))}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

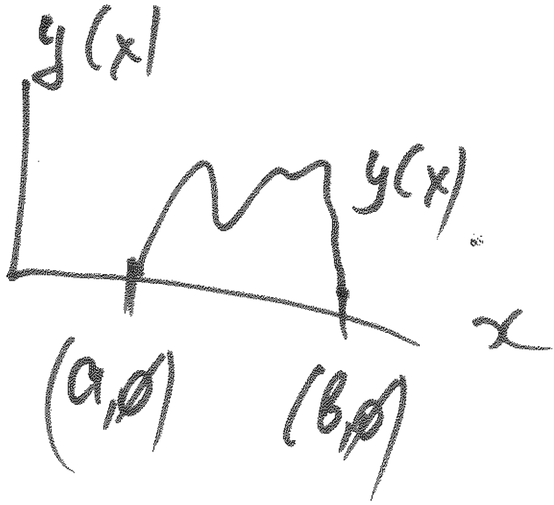
$$\frac{d F\left(f(x), f'(x)\right)}{dx} = \frac{\partial F}{\partial f} f' + \frac{\partial F}{\partial f'} f''$$

$$\frac{dB}{dx} = \frac{\partial F}{\partial f} f' + \frac{\partial F}{\partial f'} f'' - f''\,\frac{\partial F}{\partial f'} - f'\,\frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right)$$

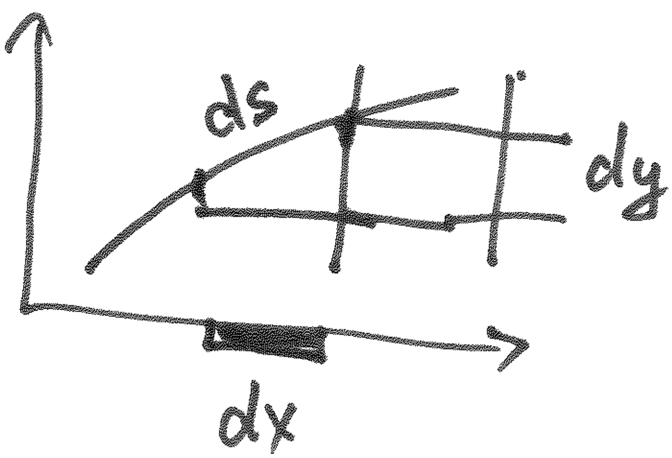
$$\frac{dB}{dx} = f'(x)\underbrace{\left(\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right)\right)}_{\text{Lagrange Euler}} = 0$$

$$B = F(f(x), f'(x)) - f'(x)\frac{\partial}{\partial f'}F(f(x), f'(x))$$

$$= \text{const}$$



$$y(a) = y(b) = 0$$



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= dx\sqrt{1 + (y'(x))^2}$$

$$L(y(x)) = \int_a^b dx\sqrt{1+(y'(x))^2} = L(y(x), y'(x))$$

$$F(y'(x)) = \sqrt{1+(y')^2}$$

$$0 = \frac{\delta L(y(x), y'(x))}{\delta y(x)} \Rightarrow$$

$$\cancel{\frac{\partial}{\partial y(x)}}\ \frac{d}{dx}\ \frac{\partial F}{\ }$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = \emptyset$$

$$\frac{\partial F}{\partial y} = \emptyset$$

$$F(y') = \sqrt{1+(y')^2}$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \emptyset$$

$$\frac{\partial F}{\partial y'} = C = \frac{2y'}{2\sqrt{1+(y')^2}} = C$$

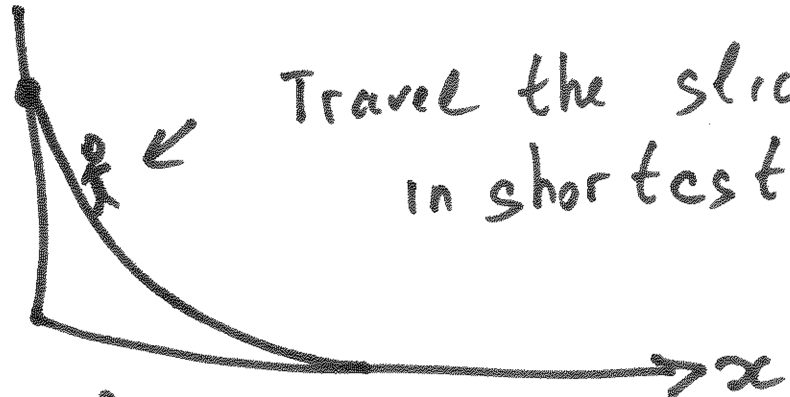
$$y' = C\sqrt{1+(y')^2}$$

$$(y')^2 = C^2(1+(y')^2)$$

$$= C^2 + C^2(y')^2$$

$$(y')^2(1-C^2) = C^2 \Rightarrow \left[y'(x)\right]^{-1} = \frac{\sqrt{1-c^2}}{c}$$

$$L = \int_a^b F(y) \sqrt{1 + (y')^2}\, dx$$



Travel the slide
in shortest slide

Boston
cop's
slide

$$E = \frac{mv^2}{2} + U(x) \leftarrow \text{Energy}$$

$$= \frac{mv^2}{2} + mg\,y(x) = E$$

$$v = \sqrt{2\left(E - mg\,y(x)\right)/m}$$

$$v(y(x)) = \sqrt{2\left(E - mg\,y(\cancel{x})\right)/m}$$

$$= \sqrt{2\frac{E}{m} - 2g\,y(x)}$$

$$v(y(x)) = \sqrt{2E - 2y(x)} \qquad m = g = 1$$

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$$S = v \cdot t \Rightarrow t = s/v$$

element of length

$$T[y(x)] = \int \frac{ds}{v}$$

$$= \int dx \frac{\sqrt{1+(y'(x))^2}}{\sqrt{2(E-y(x))}}$$

$$B = F - y' \frac{\partial F}{\partial y'} = C.$$

$$\frac{\sqrt{1+(y')^2}}{\sqrt{2(E-y(x))}} - y' \frac{y'}{\sqrt{2(E-y)(1+y'^2)}} = C$$

$$\frac{1+(y')^2-(y')^2}{\sqrt{2(E-y)(1+y'^2)}} = C$$

$$2\left(E - y(x)\right)\left(1 + \left(y'(x)\right)^2\right) = 1/C^2$$ 9

$$x(\theta) = x_0 + a(\theta - \sin\theta)$$

$$y(x) = y_0 - a(1 - \cos\theta)$$

BRACHISTochone

rope between 2 fixed points

$y(x)$

$$L = \int_a^b \sqrt{1+y'^2(x)}\,dx$$ - Fixed constrain

Potential energy

$$\underset{\text{Functional}}{V[y(x)]} = \underbrace{ga}_{=1}\int_a^b \underset{\text{length}}{y(x)}\,\underset{\text{length}}{ds} = \int_a^b y(x)\sqrt{1+y'^2}\,dx$$

$$F(y(x), y'(x)) = V[y(x)] + \lambda \cdot L$$

$$= (y(x)+\lambda)\sqrt{1+(y')^2}$$

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$I[y(x)] =$

$$= \int dx \left[y(x)\sqrt{1+y'^2(x)} - \lambda\sqrt{1+y'^2} \right]$$

$\hookrightarrow \quad y(x) = g(x, C, \lambda)$

$$F - y'\frac{\partial F}{\partial y'} = C$$

$$(y(x)-\lambda)\sqrt{1+y'^2} - \frac{(y')^2(y-\lambda)}{\sqrt{1+y'^2}} = C$$

$\Downarrow \quad \cdots$

$$y(x) = C\cosh\left(\frac{x-x_0}{C}\right) - \lambda$$

One particle

velocity

$$L = L(\underline{q}(t), \dot{\underline{q}}(t), t)$$

position

For now:

$$L = L(\underline{q}(t), \dot{\underline{q}}(t))$$

$$dL(\underline{q}(t), \dot{\underline{q}}(t)) = \frac{\partial L}{\partial \underline{q}}\, d\underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}}\, d\dot{\underline{q}}$$

$\underline{q}(t)$ - "generalized" coordinate

Definition:

Generalized momentum $p(t) \equiv \frac{\partial L}{\partial \dot{\underline{q}}}$

$$\frac{\partial L}{\partial \underline{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \emptyset \Rightarrow \frac{\partial L}{\partial \underline{q}} = \dot{p}$$

$$dL = \frac{\partial L}{\partial q}\, dq + \frac{\partial L}{\partial \dot q}\, d\dot q$$

$$= \dot p\, dq + p\, d\dot q$$

$$d(p\dot q) = \dot q\, dp + p\, d\dot q \Rightarrow p\, d\dot q = d(p\dot q) - \dot q\, dp$$

$$dL = \dot p\, dq + d(p\dot q) - \dot q\, dp$$

$$d(L - p\dot q) = \dot p\, dq - \dot q\, dp$$

$$d(\underbrace{p\dot q - L}_{H}) = \boxed{\dot q\, dp} - \dot p\, dq$$

$$H \equiv H(q,p) \qquad p = \frac{\partial L}{\partial \dot q} = p(q,\dot q)$$

$$dH(q,p) = \frac{\partial H}{\partial q}\cdot dq + \boxed{\frac{\partial H}{\partial p}\, dp}$$

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$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$H \equiv H(p, q)$$

$$H = H(p(t), q(t))$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial p} \cdot \dot{p} + \frac{\partial H}{\partial q} \cdot \dot{q} =$$

$$= -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = \emptyset$$

- $L = T - U = \dfrac{m\dot{x}^2}{2} - V(x)$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$-\frac{\partial V}{\partial x} - m\ddot{x} = 0$$

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

- $p = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$; $\dot{x} = p/m$

$$H(p,q) = p\dot{q} - L$$

$$H(p,x) = p\dot{x} - L\Big|_{\dot{x}=p/m} = \frac{p^2}{m} - \frac{m}{2}\left(\frac{p^2}{m^2}\right) + V(x)$$

$$= \frac{p^2}{2m} + V(x)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\dot{x} = \frac{\partial H}{\partial p} = p/m$$