MATH 2400
INTRODUCTION TO DIFFERENTIAL EQUATIONS

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Tuesday-Friday 8:30-9:50 **AE214**
Office Hours : Monday 18-20, Webex meeting room lvovy
yurilvov.com/DFQ Lecture Notes *yurilvov.com/DFQFall2023.pdf*[*]
TA: Miles Corn, cornm@rpi.edu Office Hours Fr 4-6pm AE432

August 28, 2023

Midterms are Sep 29, Oct 31, Dec 1, 2023
No Lecture on Nov 28, 2023. Make up lecture Nov 15, 2023

If you are late to class, you agree to either dance, tell a joke or sing.

1 Outline

First-order differential equations, second-order linear equations, eigenvalues and eigenvectors of matrices, systems of first-order equations, stability and qualitative properties of nonlinear autonomous systems in the plane, Fourier series, separation of variables for partial differential equations.

Prerequisites: MATH 1020 and some knowledge of matrices.

2 Learning Outcomes:

To learn the basics of differential equations (as described in the outline and recommended problems) which are a crucial tool in higher-level scientific and engineering subjects. An additional benefit will be to improve mathematical manipulation, modeling, and reasoning skills.

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[*] *not guaranteed*

Table 1: Outline

Week	Pages	Title	Sections	Homework Problems
1	13	Intro/First Order DFQ, Separable Eqn	1.1,1.2, 2.1	1.2.3-4 2.1.1-2 2.1.6
2	24	Integrating Factor and Modeling	2.2, 2.3	2.2.1-3 2.3.2-3 2.3.8
3	40	Steady State and Stability	2.4	2.4.2-6, 2.4.10
4	48	Second Order Linear Equation	3.1-3.5	3.2.4-5, 3.5.3-4,3.5.6-7
5	60	Inhomogeneous Eqn and MethUndetCoef	3.6-3.8	3.8.1-2, 3.8.4-5
6	76	Variation of a Parameter, Mechanical Vibrations, Euler Eq.	3.9-11	3.9.1-4,3.10.1-12,3.11.1-2
7	98	Linear Systems, General Solutions of HomogEqn, Review, Solving	4.1-4.4	4.1.1-3,4.3.1-4,4.5.1-3
8	108	Phase Plane Technique and Stability	4.6 4.7	4.6.1-2, 4.7.1-4
9	122	Nonlinear Systems and Stability	5.1 5.2	5.1.1-2, 5.2.1-2
10	138	Periodic Solutions and Central Force Field	5.3 5.4	5.3.1-3

3 Course Outline

- Chapter 1, Introduction
- Chapter 2, First Order Equations:
  Separable Equations, Integrating Factor, Modeling, Phase Plane Technique
- Chapter3. Second Order Equations:
  IVP, Homogeneous Equations, Inhomogeneous equations, Method of Undetermined Coefficients, Variations of a Parameter, Oscillations, Euler equation
- Chapter 4 Linear Systems: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$.
- Chapter 5 Nonlinear Systems
- Chapter 6 Laplace Transforms:
  definition, inverse Laplace Transform, solving the ODE with Laplace transform
- Chapter 7 Partial Differential Equations:
  separation of variables, Sine and Cosine transforms, Wave Equations, BVP

$$\underline{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A \in R^{n \times n}$$

Here the problem are marked by a section number followed by a problem number. For example 2.2.1-3 means problems one, two and three in section 2.2.

4 Grade Policy, Homework, Exams

Take all your quizzes, remove the worst one, average the result.

Take your three midterms and quiz averages computed above, remove the worst number, average the result.

Take the resulting number, multiply it by 0.7 and add your final, multiplied by 0.3.

Alternatively, take the resulting number, multiply it by 0.8 and add your final, multiplied by 0.2.

Your numerical grade is the larger of the two numbers.

Extra Credit : Final Project, 10 percent, limited availability.

I will assign a grade based on the following:

$90 \leq g \leq 100$ is A
$80 \leq g < 90$ is B
$70 \leq g < 80$ is C
$64 \leq g < 70$ is D
$g < 66$ is F

Grades modifiers will be used: $0 \leq g \leq 3$ is "-", $3 < g \leq 7$ has no modifier, $7 < g$ is "+". Homeworks and exams may contain problems for extra-credit.

There is no "make up" policy for homeworks or exams.

5 Few Additional Notes

5.1 Homeworks, attendance

Solutions to homework problems are not provided.

There is no "curve grading".

Attendance may be be logged.

There is no requirement to attend class. This said, however, long-time experience shows that students who do not attend class and/or recitations usually do poorly in the course. Neither the instructor nor the teaching assistant are in any way responsible for briefing students who missed class on the missed material and/or announcements.

you are responsible for attending quizzes

Usage of **LAPTOPS, CELL PHONES, ipods or lightsabers.** will result in 5 percent substracted from your final grade.

My promise: Those who will pass the course will know Differential Equations

Academic Integrity The grade you receive for the course will be based on the work that you do.

With this principle in mind, the work (exams, homework, computer programs) that you present for a grade MUST, in fact, be of your own.

With respect to the exams, this means that no assistance or collaboration of any kind is permitted (other then assisace obtained from the instructor). Anyone violating this policy will receive an exam grade of zero and will be reported to the Dean of Students.

With respect to problem sets, you are free to seek assistance or advise from any person, book or computer. However, what you hand in must be your own work. Violating of this policy will result in a score of zero for the assignment and will be reported to the Dean of Students.

Second violation will get you grade F for the course.

Note that your health, need of financial aid, need to maintain GPA, need to graduate, obtain employment, etc **will not** be considered.

If you have serious problems when trying to solve the suggested homeworks, please seek tutoring from the Advising and Learning Assistance Center.

5.2 Drop In Tutoring

Advising and Learning Assistance Center holds free drop-in tutoring for Differential Equation Class. Contact Sharon McGrath, mcgras@rpi.edu.

6 Books

Introduction to Differential Equations 2e Second Edition, Mark Holmes,
ISBN-10 1975077202 ISBN-13 978-1975077204

7 Final Project

Optional final project is consists of inclass 15 minutes presentation and a final written report on the topic of your choice where you use methods of differential equation to solve some cool problem.

Time line for final project:

- ***stage one*** *- you think about what you want to do, talk to your friends, talk to me, bounce ideas of the wall -* ***September 7, 2023****.*

- **stage 2**- you submit to me a title and a one page description of your project **September 15, 2023** to me.
- **stage 3** I choose 3 or 4 best projects and will notify you if you are selected.
- **stage 4** - you present to me your talk October 15, 2023
- **stage 4** you give in-class presentation and answer questions November 15, 2023
- Last day of classes - you give me your written report in LATEX format December 8, 2023

Submission of your final report constitutes your agreement to wave all the copyrights of your report. It may be made available to other DFQ students. To be eligible for a final project, your grade with out final project should be C or above

WELCOME

A

DFQ

L1

Yuri Lvov

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Monday 6-8 pm webex
lvovy

lvovy@RPI.edu

3 midterms + Quizes

4 numbers

MAX[~~lowest grade~~

$0.7\,(\text{Average of these 3 \#})$

$+\ \text{Final}\cdot 0.3$

or ?

$$0.8\left(\frac{\#+\#+\#}{3}\right)+0.2\cdot\text{Final}\Big]$$

2nd order

ODE

C

$$y''(x) + y(x) = 0$$

↳ y[illegible] - dependent variable

x - independent variable

$$y''(x) \equiv \frac{d^2 y(x)}{dx^2}$$

ODE ✓

ordinary differential equation

1 independent variable

PDE

Partial differential equation

2 or mor

D

PDE

$u(x, t)$ → time

coordinate

Heat equation

first order in t

second order in X

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

short hand

$$u_t = u_{xx}$$

E

number of
dependent variables

1
single
equation

2 or
more
system
of DE

- order of differential equation

order of the highest
derivative

First order:

$$\frac{dy(x)}{} = g(x)$$

F

Linear

versus

~~nonlinear~~

nonlinear

$(y(x))^n, \; n \neq 1$

$y'(x) = \sin(y(x))$

a bit

dependent variable is linear: $y(x)$

$$y'''(x) + y''(x) \cdot x^2 + y'(x) + \sin(x) = 0$$

G

homogeneous

$y(x) \equiv 0$

is a

sltn

versus

nonhomogeneous

$\downarrow$

$y(x) \equiv 0$

is NOT

a

solution

Linear equations

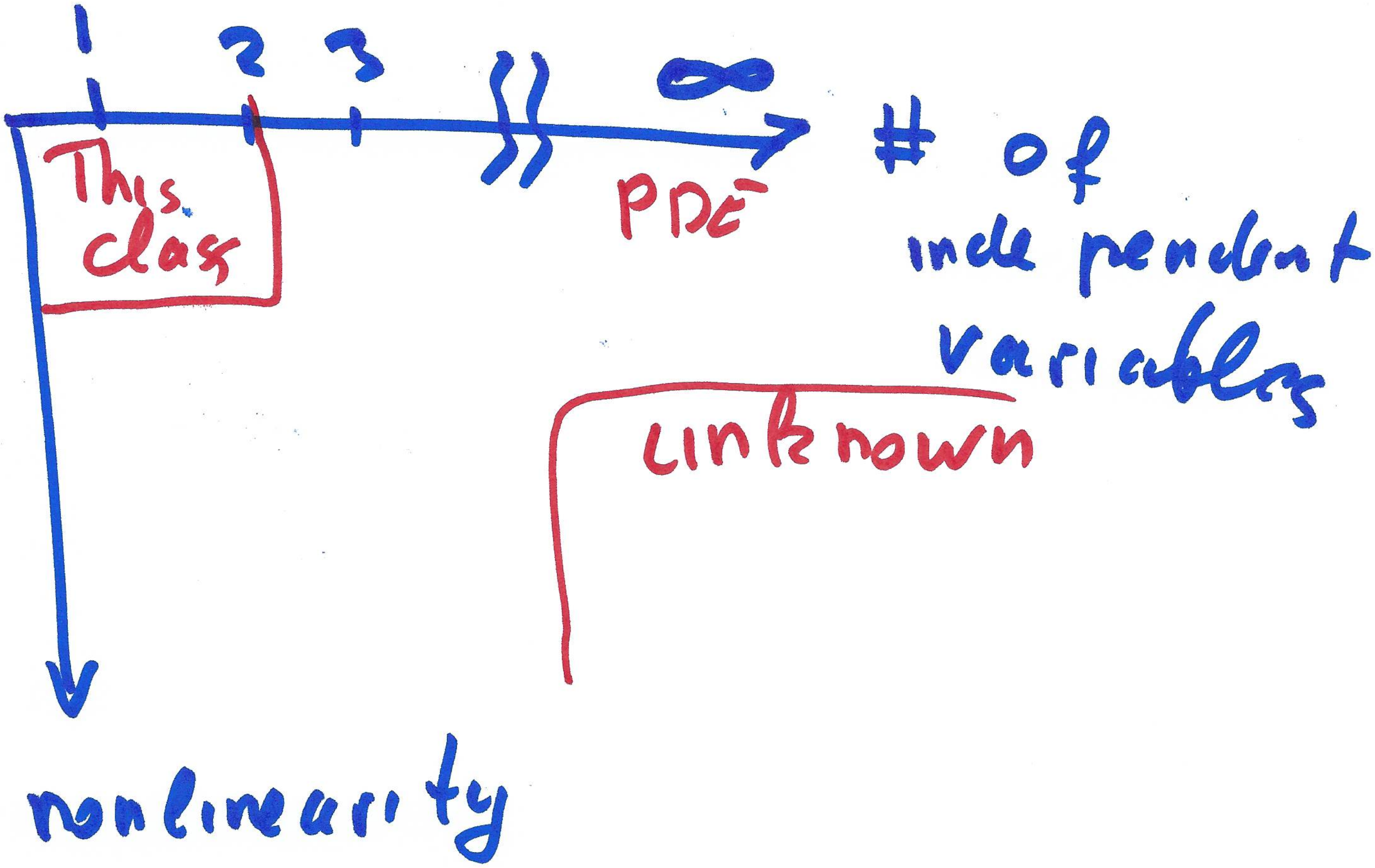
~~$y''(x) + \sin(y($~~

$y''(x) + x^2 y(x) = 0$

$y''(x) + y(x) =$

$= \sin(x)$

F


1
2
3
∞
# of independent variables
This class
PDE
unknown
nonlinearity

G

$$y''(x) + y(x) = 0$$

Solution: is an expression for $y(x)$ which makes eqn to equality

$$y(x) = A \cdot \sin(t + \varphi)$$

A, φ are arbitrary constants

$$y'(x) = A \cdot \cos(t + \varphi)$$

$$y''(x) = -A \sin(t + \varphi)$$

$$y'' + y \overset{\checkmark}{=} 0$$

LMS

Chapter 2

First order ODE

$$\frac{dy(x)}{dx} = F(y(x), x)$$

NO general analytical solution

$$y'(x) = \sin(y(x))\cdot(x^2+1)$$

- Separable ODE of 1st order

$$F(y(x),x) = f(x)\cdot g(y(x))$$

- Autonomous 1st order ode:

$$\frac{dy(x)}{dx} = f(y(x));$$

phase plane

K

Separable equations

- Assume $g(y) \neq 0$
- non unique

$$\left[\frac{d}{d\cancel{x}} y(x) = f(x)\, g(y(x)); \right] \quad \frac{dx}{g(y(x))}$$

$$\frac{d}{dx} y(x) = f(x)\, g(y(x)) \qquad \times \frac{dx}{g(y(x))}$$

$$\int \frac{d\,y(x)}{g(y(x))} = \int f(x)\, dx$$

implicit solution

express $y(x)$

explicit solution

example

L

$$\boxed{y(x) = \frac{1}{\sqrt{x+C}}}$$

$$\frac{2\,dy(x)}{dx} = -\,y^3(x);$$

First order nonlinear ode.

$$y(x) = 0;$$

arbitrary constant

$$-2\int\frac{dy}{y^3} = +\int dx = x + C$$

$\|$

$(y)^{-2}$

$$\frac{1}{y^2(x)} = x + C$$

$$\frac{d}{dx}\left(y^{-2}\right) = \frac{d}{dx}\frac{1}{y^2} = -2y^{-3}$$

$$= -2/y^3$$

M

$$\left\{\begin{array}{l} 2y'(x) = -y^3(x) \Rightarrow \begin{array}{l} y(x)=0 \\ y(x)=\dfrac{1}{\sqrt{x+c}}\end{array} \\ y(x=1)=1 \leftarrow \end{array}\right.$$

Initial condition (IC)

Initial value problem

Satisfy IC:

$$y(x=1)=\frac{1}{\sqrt{1+c}}=1 \Rightarrow c=0$$

$$y(x)=\frac{1}{\sqrt{x}}$$

N

Example

$$y'(x) = \frac{y(x)}{1-y(x)};$$

$$y(0)=1.$$

$$\frac{dy(x)}{dx} = \frac{y(x)}{1-y(x)}$$

$$\frac{dy(x)}{y(x)}(1-y(x)) = dx$$

$$\boxed{\begin{array}{l} \ln 1 - 1 = 0 + C \\ C = -1 \end{array}}$$

$$= \int \frac{dy(x)}{y(x)} - \int dy(x) = \int dx = x + C$$

$$= \ln(y(x)) - y(x) = x + C$$

implicit solution

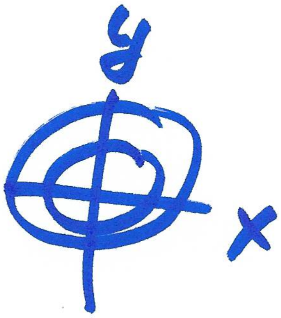
$$y(x) = \ldots$$

Example

$$\begin{cases} \dfrac{dy(x)}{dx} = -\dfrac{x}{y} \\ y(0)=1 \end{cases}$$

IVP

$$y\,dy = -x\,dx$$



$$\int y\,dy + \int x\,dx = 0$$

$$y^2 + x^2 + C = 0$$

$$C = -R^2$$

$$\boxed{\begin{array}{c} 1 = \pm\sqrt{R^2} \\ \text{choose } + \\ R = 1 \end{array}}$$

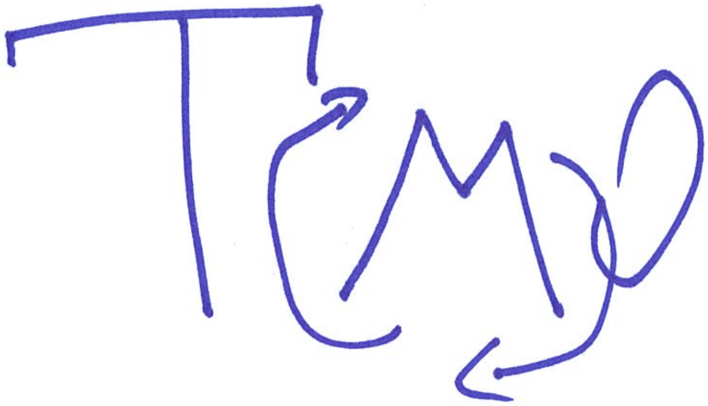
$$y^2 + x^2 = R^2$$ - implicit solution

$$y(x) = \pm\sqrt{R^2 - x^2}$$ explicit

A

L2

LectuЯƎ



First order ODE

Separation of variable

$$y'(x) = F(x)\, G(y(x))$$

$$\int \frac{dy}{G(y(x))} = \int F(x)\,dx$$

Linear first order ordinary differential equations

$$\frac{dy(x)}{dx} + p(x)\,y(x) = g(x)$$

The most general form.

- Simplest case:

  $p(x) \equiv 0 \Rightarrow \dfrac{dy(x)}{dx} = g(x)$

  $y(x) = \int g(x)\,dx$;

- Another limit $g(x) = 0$

$$\int \frac{dy(x)}{y(x)} = -\int p(x)\,dx$$

C

Integrating factor:

$\mu(x)$: function 2B found:

$$\mu(x)\frac{dy(x)}{dx} + p(x)y(x)\mu(x) = g(x)\mu(x)$$

Choose $\mu(x)$ so the LHS is

$$\frac{d}{dx}\left[\mu(x)y(x)\right]$$

Require: $\frac{d}{dx}(\mu(x)y(x)) =$

$\underline{\mu(x)}$

$$= \mu(x)\frac{dy}{dx} + p(x)y(x$$

$$\mu'(x)y(x) + \underline{\mu(x)y'(x)} = \underline{\mu(x)y'(x)} + \underset{\boxed{\mu(x)}}{p(x)}y(x$$

$$\frac{d\mu(x)}{dx} = p(x)\mu(x)$$

$$\frac{d}{dx}\Big(f(x)\,g(x)\Big) =$$

$$= \lim_{\underbrace{\Delta \to 0}} \frac{f(x+\Delta)\,g(x+\Delta) - f(x)\,g(x)}{\Delta}$$

$$= \frac{\big(f(x) + f'(x)\Delta\big)\big(g(x) + g'(x)\Delta\big) - f(x)\,g(x)}{\Delta}$$

$$= \lim_{\Delta \to 0} \Big(\underline{f(x)\,g(x)} + f'(x)\,g(x)\Delta + f(x)\,g'(x)\Delta + f'(x)\,g'(x)\Delta^2 \underline{- f(x)\,g(x)}\Big)/\Delta$$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

$$+ \lim_{\Delta \to 0} \Delta\, f'(x)\,g'(x)$$

E

$$\frac{d\mu(x)}{dx} = p(x)\mu(x)$$

$$\int \frac{d\mu(x)}{\mu(x)} = \int p(x) \, dx$$

$$\ln \mu(x) = \int p(x) \, dx$$

$$\boxed{\mu(x) = e^{\int p(x) dx}}$$

$$\left[\frac{dy(x)}{dx} + p(x)y(x) = g(x)\right]e^{\int p(x)dx} \quad F$$

$$e^{\int p(x)dx}y'(x) + p(x)y(x)e^{\int p(x)dx}$$

$$e^{\int p(x)\,dx}$$

$$\frac{dy(x)}{dx} + p(x)\,y(x)\,e^{\int p(x)dx} = g(x)\,e^{\int p(x)dx}$$

$$\frac{d}{dx}\left[e^{\int p(x)\,dx}\,y(x)\right] = g(x)\,e^{\int p(x)dx}$$

$$\mu(x) = e^{\int p(x)dx} = e^{P(x)+C}$$

H

Example

$$\left.\begin{array}{l} y'(x) - 2x\, y(x) = x \\ \qquad y(0) = 0 \end{array}\right\} \text{Initial value problem}$$

Sltn

Solution

$$\mu(x) = e^{\int p(x)dx} = e^{-2\int x dx} = e^{-x^2}$$

$$\boxed{p(x) = -2x}$$

$$\underbrace{y'(x)e^{-x^2} - 2x\, y(x)e^{-x^2}} = x e^{-x^2};$$

$$\frac{d}{dx}\left(y(x)e^{-x^2}\right) = x e^{-x^2}$$

$$y(x)e^{-x^2} = \int x e^{-x^2} dx = \int e^{-x^2} \frac{dx^2}{2} =$$

$$= \int e^{-u} du \cdot \frac{1}{2} = -\frac{1}{2} e^{-x^2} + C$$

$$u = x^2$$

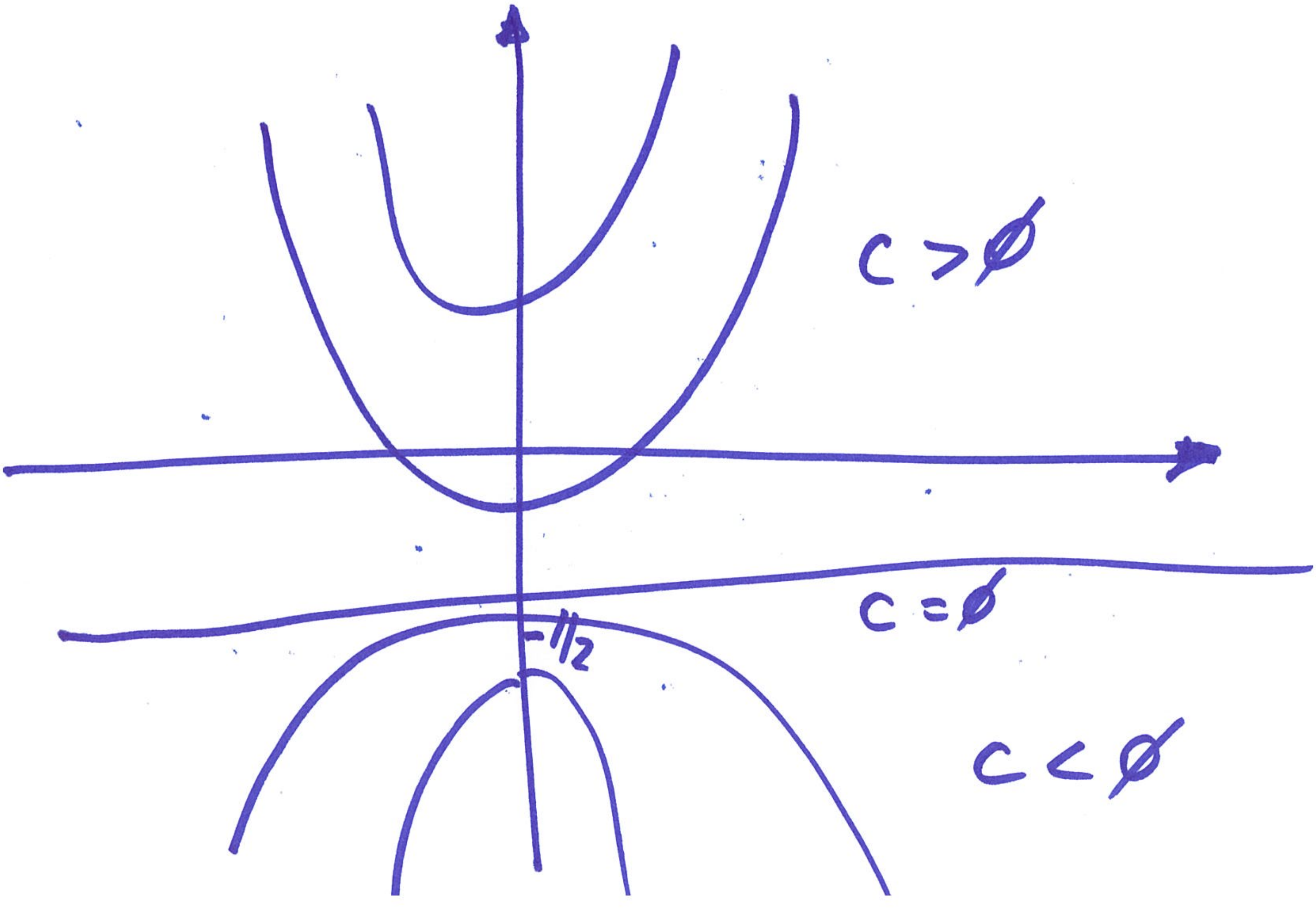
$$y(x)e^{-x^2} = -\frac{1}{2}e^{-x^2} + C.$$ I

$$y(x) = -\frac{1}{2} + Ce^{x^2}$$

$$y(0) = 0$$

$$y(0) = -\frac{1}{2} + C = 0 \Rightarrow C = 1/2$$

$$y(x) = \left(e^{x^2} - 1\right)/2$$



Example

① $y' + 2y = e^{-x}$

② $\mu(x) = e^{\int 2dx} = e^{2x}$

③ $\left(y'(x)e^{2x} + 2y(x)e^{2x}\right) = e^{x}$

④ $\frac{d}{dx}\left(y(x)e^{2x}\right) = e^{x}$

⑤ $y(x)e^{2x} = \int e^{x}dx = e^{x} + C$

⑥ $y(x) = e^{-x} + Ce^{-2x}$

K

$$t\frac{d}{dt}y(t) + 2y(t) = 4t^2$$

$$y(1) = 2;$$

Integrating factor: $p(x) = \frac{2}{t};$

~~$\mu(x) =$~~

$$\frac{d}{dt}y(t) + \frac{2}{t}y(t) = 4t;$$

$$\boxed{y(t) = t^2 + c/t^2}$$

~~$\mu(x) =$~~

[illegible]

$$\mu(t) = e^{\int\frac{2\,dt}{t}} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2\frac{d}{dt}y(t) + 2t\,y(t) = 4t^3$$

$$\frac{d}{dt}\left(t^2y(t)\right) = 4t^3;\ t^2y(t) = \int 4t^3\,dt =$$

$$= t^4 + C$$

$y(t) = t^2 + c/t^2$

L

$y(1) = 1 + c = 2$

$c = 1$

$y(t) = t^2 + 1/t^2$

Example

M

$$y'(x) + 2y(x) = xe^{-2x}$$

$$\mu(x) = e^{2x}$$

$$\frac{d}{dx}\left(y(x)e^{2x}\right) = xe^{-2x}\cdot e^{2x} = x$$

~~$y(x)e^{2x} = \int xe^{-2x}dx =$~~

~~$= \int x\,de^{-2x}\cdot\left(-\frac{1}{2}\right)$~~

$$y(x)e^{2x} = \int x\,dx = x^2/2 + C$$

$$y(x) = \frac{x^2e^{-2x}}{2} + Ce^{-2x}$$

$y'(x) - 2x\,y(x) = 1$ N

$\mu(x) = e^{\int p(x)dx} = e^{-\int 2x dx}$

$= e^{-x^2}$

$$\frac{d}{dx}\left(y(x)e^{-x^2}\right) = e^{-x^2}$$

$$y(x) = e^{x^2}\int e^{-x^2}dx$$

No explicit solution

$$\frac{d}{dx}\left(F(x)\right) = e^{-x^2}$$

$$\left[y'(x) + p(x)\,y(x) = g(x)\right] e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(y(x)\, e^{\int p(x)dx} \right) = g(x)\, e^{\int p(x)dx}$$

indefinite integrals

$$\mu(x) = e^{\int_0^x p(x)dx}$$

§2.4 Modeling

$$\boxed{\begin{aligned} \frac{d}{dt} N &= r N \\ r &\to r(1-N) \end{aligned}}$$

Rate Process = + Add

Radioactive decay

The rate of decay of radioactive is proportional to the amount of material

$Q(t)$ - amount of material

$$\frac{d\,Q(t)}{dt} = -r\;Q(t)$$

A

L3

Modeling

13

$Q(t)$ - mass of radioactive material

Speed of decay is $\sim$ to the mass

proportional

$\frac{d}{dt} Q(t) = \cdot$

C

$$\frac{d}{dt} Q(t) = - r Q(t); \quad r > 0$$

→ Violent gang $N(t)$
speed of killing is
~ to the amount
of people

$$\frac{d}{dt} N(t) = - r N(t); \quad r > 0$$

→ population grow ~
to amount of people

$$\frac{d}{dt} N(t) = r N(t);$$

D

$N(t)$ - characteristic

$$\frac{d}{dt} N(t) = \text{In} - \text{Out};$$

→ $T(t)$ - temperature

T_a - ambient temperatur

$$\frac{d}{dt} T(t) = -k\left(T(t) - T_a\right)$$

$k > 0$

Newton law of cooling

E

Radioactive material

$$\frac{d}{dt} Q(t) = - r \, Q(t), \quad r > 0;$$

$$\ln Q = \int \frac{dQ(t)}{Q(t)} = - r \int dt = - r t + C$$

$$Q(t) = e^{-rt+C} = \tilde{C} e^{-rt}$$

$$e^{C} \equiv \tilde{C}$$

$$Q(t=0) = Q_0 = \tilde{C} e^{-rt}\Big|_{t=0} = \tilde{C}$$

$$\tilde{C} = Q_0 \Rightarrow \boxed{Q(t) = Q_0 e^{-rt}}$$

F

$Q(t) = Q_0 e^{-rt}$;

Problems

- $Q_0 > 0$; T - wait ; $Q_1 < Q_0$

what is rate ?

$$Q(t=T) = Q_0 e^{-rT} = Q_1$$

$$e^{-rT} = Q_1/Q_0$$

$$-rT = \ln Q_1/Q_0$$

$$r = -\frac{1}{T} \ln \frac{Q_1}{Q_0}$$

A

L3

Modeling

$Q(t)$ - mass of radioactive material

Speed of decay is $\underset{\text{proportional}}{\sim}$ to the mass

$\frac{d}{dt} Q(t) = ..$

C

$$\frac{d}{dt} Q(t) = -r\,Q(t); \quad r \geq 0$$

→ violent gang $N(t)$
speed of killing is
~ to the amount
of people

$$\frac{d}{dt} N(t) = -r\,N(t); \quad r \geq 0$$

→ population grow ~
to amount of people

$$\frac{d}{dt} N(t) = r\,N(t);$$

D

$N(t)$ - characteristic

$$\frac{d}{dt} N(t) = \text{In} - \text{Out};$$

→ $T(t)$ - temperature

T_a - ambient temperatur

$$\frac{d}{dt} T(t) = -k\,(T(t) - T_a)$$

$k > 0$

Newton law of cooling

E

Radioactive material

$$\frac{d}{dt} Q(t) = -r\, Q(t), \quad r > 0;$$

$$\ln Q = \int \frac{dQ(t)}{Q(t)} = -r \int dt = -rt + C$$

$$Q(t) = e^{-rt+C} = \tilde{C} e^{-rt}$$

$$e^{C} \equiv \tilde{C}$$

$$Q(t=0) = Q_0 = \tilde{C} e^{-rt}\Big|_{t=0} = \tilde{C}$$

$$\tilde{C} = Q_0 \Rightarrow \boxed{Q(t) = Q_0 e^{-rt}}$$

F

$Q(t) = Q_0 e^{-rt}$;

Problems

- $Q_0 > 0$; T - wait ; $Q_1 < Q_0$

what is rate ?

$$Q(t=T) = Q_0 e^{-rT} = Q_1$$

$$e^{-rT} = Q_1 / Q_0$$

$$-rT = \ln Q_1/Q_0$$

$$r = -\frac{1}{T} \ln \frac{Q_1}{Q_0}$$

Half life time: τ G

How much time to wait for half of a material to decay?

$$Q(t) = Q_0 e^{-rt}$$

$$Q(\tau) = Q_0 e^{-r\tau} = Q_0/2$$

$$e^{-r\tau} = 1/2$$

$$\ln \ -r\tau = \ln 1/2 = -\ln 2$$

$$\tau = \frac{\ln 2}{r}$$

$$\ln \frac{A}{B} = \ln A - \ln B$$

H

$$\frac{d}{dt} N(t) = k N(t)$$

$$N(t) = N_0 e^{kt}$$

How much time it takes for population to triple?

$t = 0 \quad N(t=0) = N_0$

$t = T \Rightarrow N(T) = N_0 e^{rT} = 3N_0$

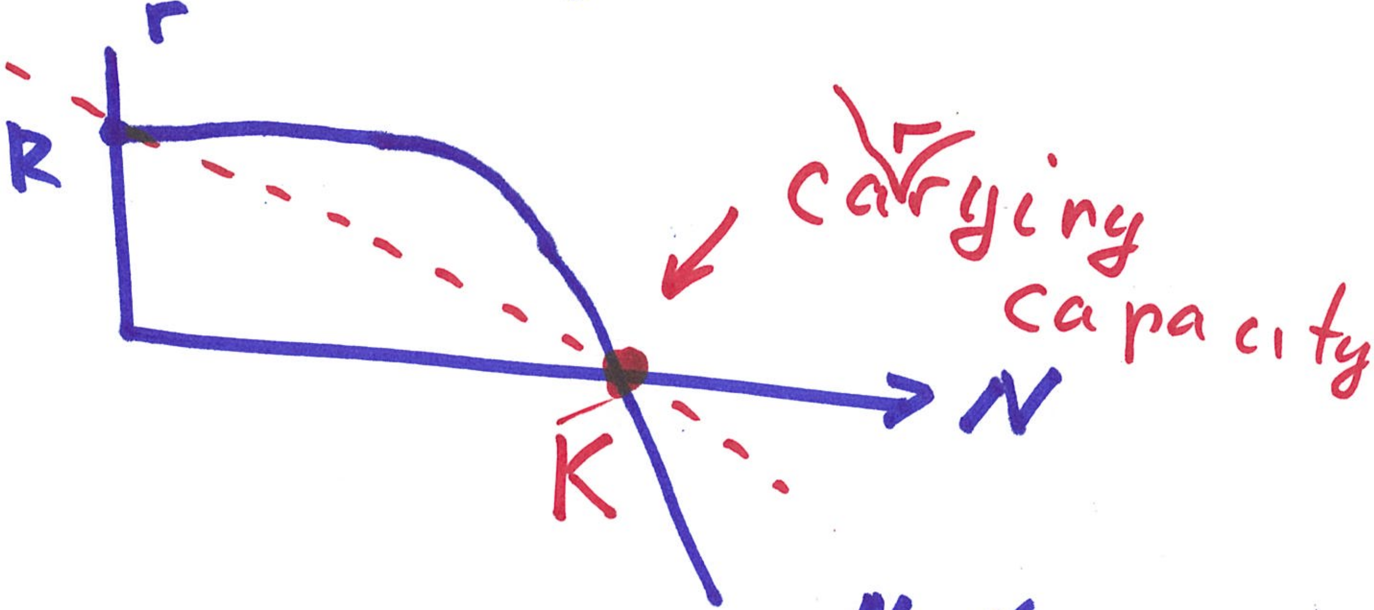
$rT = \ln 3;$

$$T = \frac{1}{r} \ln 3$$

I

$$\frac{d}{dt} N(t) = r\,N(t);$$

$$r \rightarrow r(N)$$



$N = 0 \Rightarrow r = R$

$N = K \Rightarrow r = 0$

$$r \Rightarrow r\left(1 - \frac{N}{K}\right)$$

$$\frac{d}{dt} N(t) = r\,N(t)\left(1 - \frac{N(t)}{K}\right)$$

logistic equation

$$\frac{d}{dt}N = rN\left(1-\frac{N}{K}\right);\ N \equiv N(t);$$

1st order, nonlinear,
separable

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int r\,dt = rt + C$$

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{K-N}$$

A, B are coefficients
2 b found

Method of undetermined
coefficients

$$\boxed{\frac{1}{N\left(1-\frac{N}{K}\right)}} = \frac{A}{N} + \frac{B}{K-N} = \frac{A}{N} + \frac{B/K}{1-\frac{N}{K}} \qquad K$$

$$= \cancel{\frac{A(K-N)+BN}{N(K-N)}}$$

$$= \frac{A - \frac{A \cdot N}{K} + \frac{BN}{K}}{N\left(1-\frac{N}{K}\right)} = \boxed{\frac{A + \frac{N}{K}(B-A)}{N\left(1-\frac{N}{K}\right)}}$$

$A = B = 1$

L

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int\left(\frac{1}{N} + \frac{1/K}{1-\frac{N}{K}}\right) dN =$$

$$= \int\left(\frac{1}{N} + \frac{1}{K-N}\right) dN$$

$$= \ln N - \ln(K-N) = \ln\frac{N}{K-N} =$$

$$= rt + C$$

$$\frac{N}{K-N} = e^{rt+C} ; \quad N = e^{rt+C}(K-N)$$

M

$$N\left(1+e^{rt+c}\right)= K e^{rt+c}$$

$$N = \frac{K e^{rt+c}}{1+e^{rt+c}} = \frac{K}{1+e^{-rt-c}}$$

$$N(t) = \frac{K}{1+e^{-rt}\cdot\tilde{c}}$$

$$\tilde{c} = \frac{K-N_0}{N_0}$$

$$N(t=\emptyset) = N_0 = \frac{K}{1+\tilde{c}} \Rightarrow$$

~~N(t)=~~

$$(1+\tilde{c})N_0 = K = N_0 + \tilde{c}N_0$$

N

$$\tilde{C} = \frac{K - N_0}{N_0}$$

$$N(t) = \frac{K}{1 + e^{-rt}(K - N_0)/N_0}$$

$$= \frac{K N_0}{N_0 + e^{-rt}(K - N_0)}$$

Page Newton law of COOLING
Missing

$$\frac{d}{dt} T(t) = -K\left(T(t) - \overset{\overset{\text{ambient}}{\downarrow}}{T_a}\right)$$

$$\frac{d}{dt}\left(T(t)e^{Kt}\right) = K T_a e^{Kt}$$

$$T(t) = T_a + \left(T(t) - T_a\right)e^{-Kt}$$

M

$$e^{kt}\,T(t) = \int k T_a e^{kt}\,dt$$

$$= T_a e^{kt} + C$$

$$T(t) = T_a + C e^{-kt}$$

$$T(t = 0) = T_a + C = T_o$$

$$C = T_o - T_a$$

$$\boxed{T(t) = T_a + (T_o - T_a) e^{-kt}}$$

$$\Downarrow$$

~~$T(t) = T_a(t) + (T_o$~~

given

$$\frac{d}{dt} T(t) = -k\left(T(t) - T_a(t)\right)$$

$$T(t) = T_a + (T_0 - T_a)e^{-Kt}$$ ✓

$$T(t = 2\text{ hours}) =$$

$$= T_a + (T_0 - T_a)e^{-K \cdot 2\text{ hours}}$$

$$70F + (95F - 70F)e^{-K \cdot 2\text{hours}} = 93F$$

$$= 93F$$

$$e^{-K \cdot 2\text{hours}} = \frac{93F - 70F}{95F - 70F}$$

$$K = -\frac{1}{2\text{hours}} \ln \frac{93F - 70F}{95F - 70F}$$

$$= -\frac{1}{2\text{hours}} \ln \frac{23}{25}$$

$t = 0$; $T(t=0) = 95$ F

$t = 2$ hours: $T(t = 2 \text{ hours}) = 93$F

Normal 97.6 – 99.6 F

Normal 100 F

$T_A = 70$ F

Newton law of cooling

$$\frac{d}{dt} T(t) = -K(T(t) - T_a)$$

$$\left(\frac{d}{dt} T + KT = K T_a\right) e^{Kt}$$

$$\frac{d}{dt}\left(e^{Kt} \cdot T(t)\right) = K T_a e^{Kt}$$

p

$$K = -\frac{1}{2\text{ hours}} \ln\frac{23}{25}$$

$$T(t) = T_a + (T_0 - T_a)e^{-Kt}$$

$$T(t=t^*) = T_a + (T_0 - T_a)e^{-Kt^*} = 100F$$

$$70F + (95 - 70F)e^{-Kt^*} = 100F$$

$$e^{-Kt^*} = \frac{100F - 70F}{95F - 70F}$$

$$t^* = -\frac{1}{K} \ln\frac{100-70}{95-70} =$$

$$= 2\text{ hours} \cdot \frac{1}{\ln(23/25)} \ln\frac{30}{25}$$

$$\boxed{-4\text{ hours }27\text{ min}}$$

A

L4

N15

B

Population growth

radioactive decay

$$\frac{d}{dt} N(t) = r\, N(t);$$

$$N(t) = N_0 e^{rt}$$

$r > 0$ – growth

$r < 0$ - decay

logistic model

$$\frac{d}{dt} N(t) = r N(t)\left(1 - \frac{N(t)}{K}\right);$$

Newton law of cooling

$$\frac{d}{dt} T(t) = +k(T(t) - T_a)$$

Dulong - Petit C

Nonlinear cooling

$$k \to k(T - T_a)^{1/4}$$

motivated by experime

$$\frac{d}{dt}T(t) = -k(T - T_a)^{5/4}$$

$$\int \frac{dT}{(T - T_a)^{5/4}} = -\int k\, dt$$

$$\frac{4}{(T - T_a)^{1/4}}$$

D

$$\int (T - T_a)^{-5/4}\, dT = -\int k\,dt$$

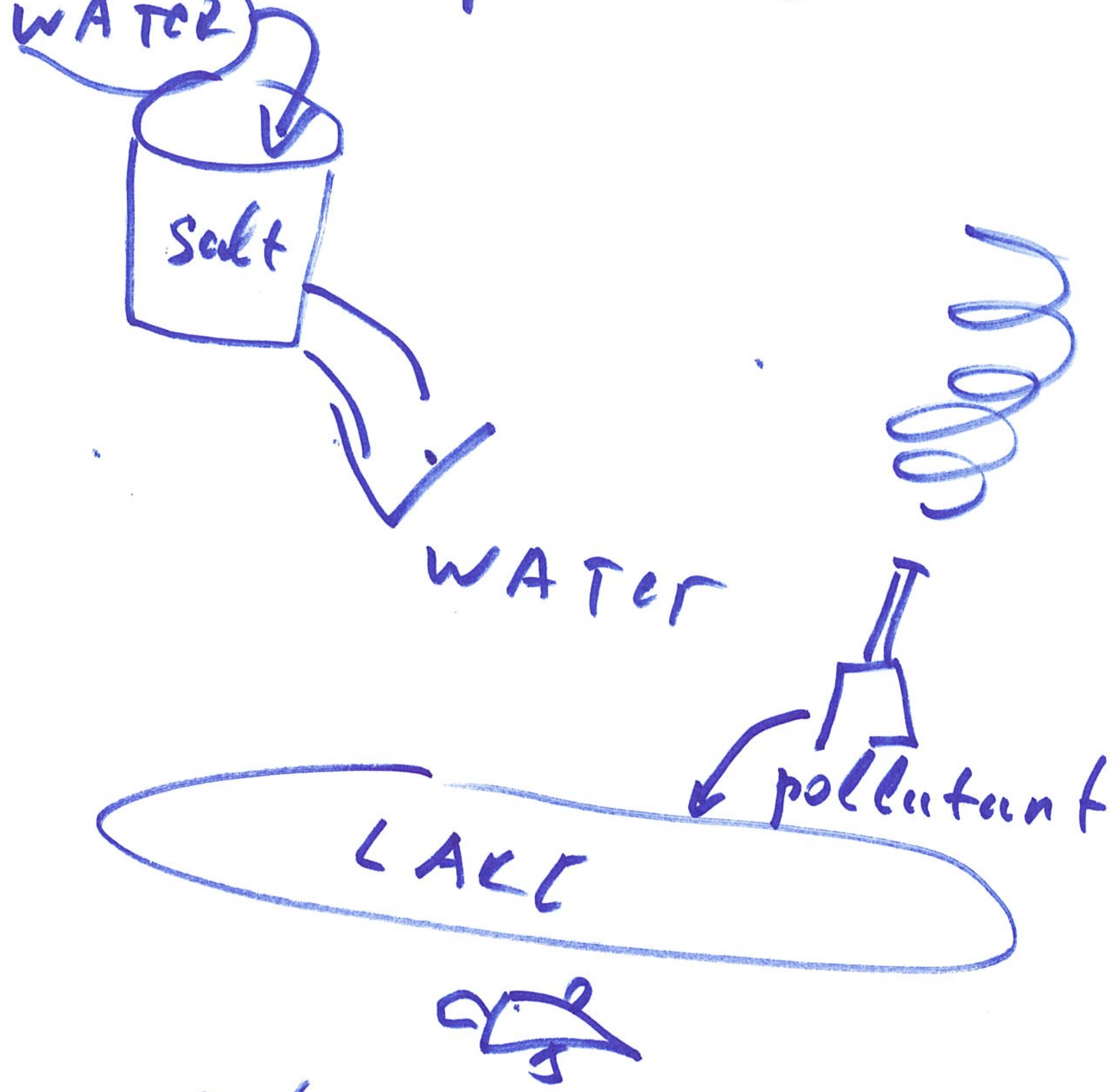
$$-4\,(T - T_a)^{-1/4} = -kt - C$$

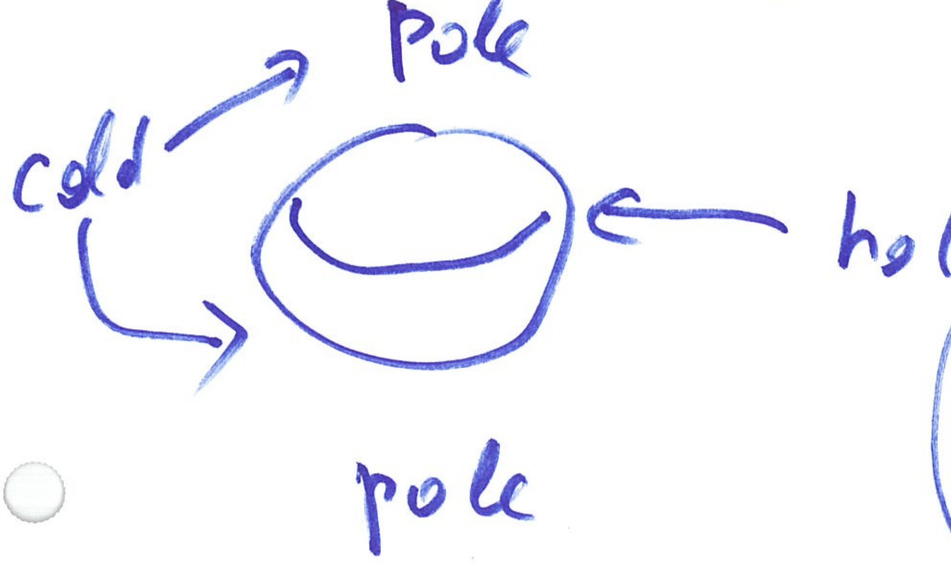
$$(T - T_a)^{-1/4} = \frac{1}{4}(kt + c)$$

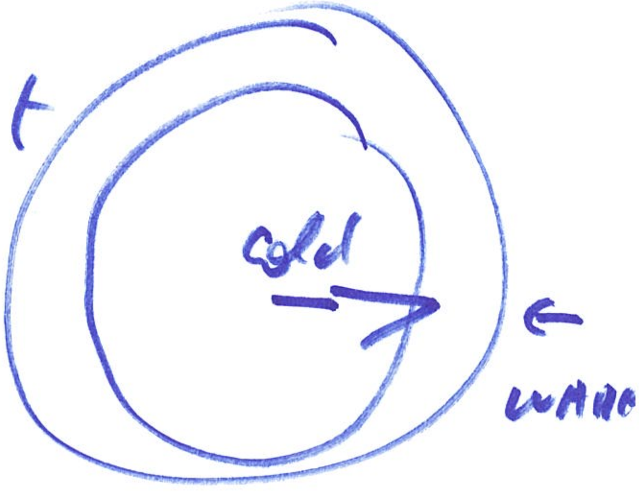
$$T(t) = T_a + \left(\frac{4}{kt + c}\right)^4$$

E

Mixing problems







← φ liters/second, C - concentrat

F

υ - volume

$Q(t)$ - amount of salt in a container at time t

$$\frac{d}{dt}Q(t) = \text{Salt IN} - \text{Salt Out}$$

$$\frac{d}{dt}Q(t) = C \cdot \varphi - \frac{Q(t)}{\upsilon} \cdot \varphi$$

concentration in grams/liter

rate in liters/second

$$\frac{d}{dt}Q(t) + \frac{\varphi}{\upsilon}Q(t) = C \cdot \varphi$$

6

$$\frac{d}{dt}\left(Q(t)e^{\frac{\psi}{\upsilon}t}\right) = c\psi e^{\frac{\psi t}{\upsilon}}$$

$$Q(t)e^{\psi t/\upsilon} = c\upsilon e^{\psi t/\upsilon} + D$$

$$Q(t) = c\upsilon + De^{-\psi t/\upsilon}$$

Initial conditions $Q(t=t_0)=Q_0$

$$Q_0 = c\upsilon + D \Rightarrow D = Q_0 - c\upsilon$$

$$Q(t) = c\upsilon + (Q_0 - c\upsilon)e^{-\psi t/\upsilon}$$

$$\boxed{Q(t) = c\upsilon\left(1 - e^{-\psi t/\upsilon}\right) + Q_0 e^{-\psi t/\upsilon}}$$

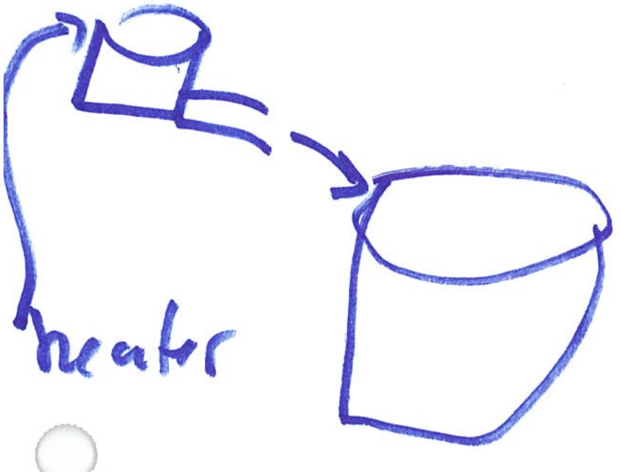
$$Q(t=0) = Q_0$$

$$\lim_{t\to\infty} Q(t) = c\upsilon$$

$$\frac{d}{dt}Q(t) = c\varphi - \frac{Q(t)}{v}\cdot\varphi$$

Sudd

$\downarrow$ $c \to c(t)$



$$\frac{d}{dt}Q(t) = c(t)\varphi - \frac{Q\varphi}{v}$$

$t = 0$

add water with contentration $\tilde{c}(t)$

Ψ liters per secont

$$\frac{d}{dt}Q(t) = c(t)\varphi - \frac{Q(t)\varphi}{v + \Psi t}$$

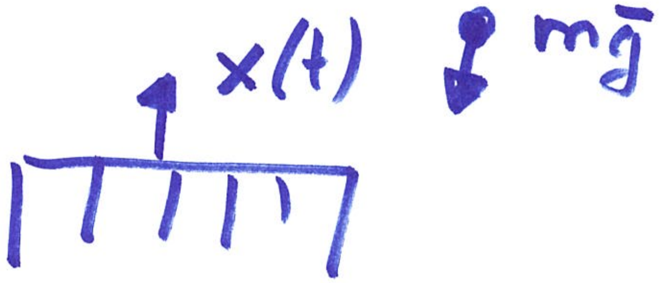
$$+ \tilde{c}(t)\Psi$$

Second Newton law

I

$$\bar{F} = m\bar{a} = m\frac{d\bar{v}(t)}{dt}$$

vertical coordinate



$$\overline{\text{drag}} \approx -\underset{\text{drag coefficient}}{c}\,\underset{\text{velocity}}{\bar{v}(t)}$$

$$F = -\underset{\text{gravity}}{mg} - \underset{\text{drag}}{cv(t)}$$

m = 5 kg; $x(t=0) = 2000$ meters

$g = 10$ meters/sec^2; $C = \frac{1}{4}$ kg/sec

- write IVP describing this scenario:

$$\begin{cases} m\dfrac{dv(t)}{dt} = -mg - c\,v(t) \\ v(t=0) = 0 \end{cases}$$

$$\frac{dv(t)}{dt} = -g - \frac{c}{m} v(t)$$

$$\frac{dv(t)}{dt} + \frac{c}{m} v(t) = -g$$

$$\left(\frac{d}{dt} v(t) e^{\frac{ct}{m}}\right) = -g e^{ct/m}$$

$$v(t) e^{ct/m} = -\frac{gm}{c} e^{ct/m} + A$$

$$v(t) = A e^{-ct/m} - \frac{gm}{c}$$

$$v(t=0) = A - \frac{gm}{c} = v_0$$

$$A = v_0 + \frac{gm}{c}$$

$$v(t) = \left(v_0 + \frac{gm}{c}\right) e^{-ct/m} - \frac{gm}{c}$$

$x(t) = ?$ K

$$\frac{dx(t)}{dt} = v(t)$$

$$x(t) = x(t=0) + \int_0^t v(t)\,dt$$

$$= x(t=0) + \int_0^t \left[\left(v_0 + \frac{gm}{c}\right)e^{-ct/m} - \frac{gm}{c}\right]dt$$

$$= x(t=0) - \left(v_0 + \frac{gm}{c}\right)e^{-ct/m} \cdot \frac{m}{c}\Bigg|_0^t - \frac{gmt}{c}$$

$$= x(t=0) - \frac{m}{c}\left(v_0 + \frac{gm}{c}\right)e^{-ct/m} + \left(v_0 + \frac{gm}{c}\right)\frac{m}{c} - \frac{gmt}{c}$$

~~$= x(t=0) - \left(\frac{mv_0}{c} + g\right)e^{-ct/m}$~~

$$x(t) = x(t=0) - \left(\frac{m v_0}{c} + g\overset{m^2/c^2}{\vee}\right)\left(e^{-ct/m} - 1\right) - \frac{g m t}{c}$$

- Terminal velocity:

$$v_{\text{Terminal}} = \lim_{t \to \infty} v(t) =$$

$$= -\frac{g m}{c}$$

Steady States and stability

$$\frac{d}{dt} x(t) = f(x(t));$$

Example

$$\frac{d}{dt} p(t) = 4(5 - p(t))\, p(t)$$

2 steady states

$$\frac{d}{dt} p(t) = 0 \Rightarrow \begin{array}{l} p(t) = 0 \\ p(t) = 5 \end{array}$$

fixed points

Modelling

:

Heating/cooling

population/radioactivity

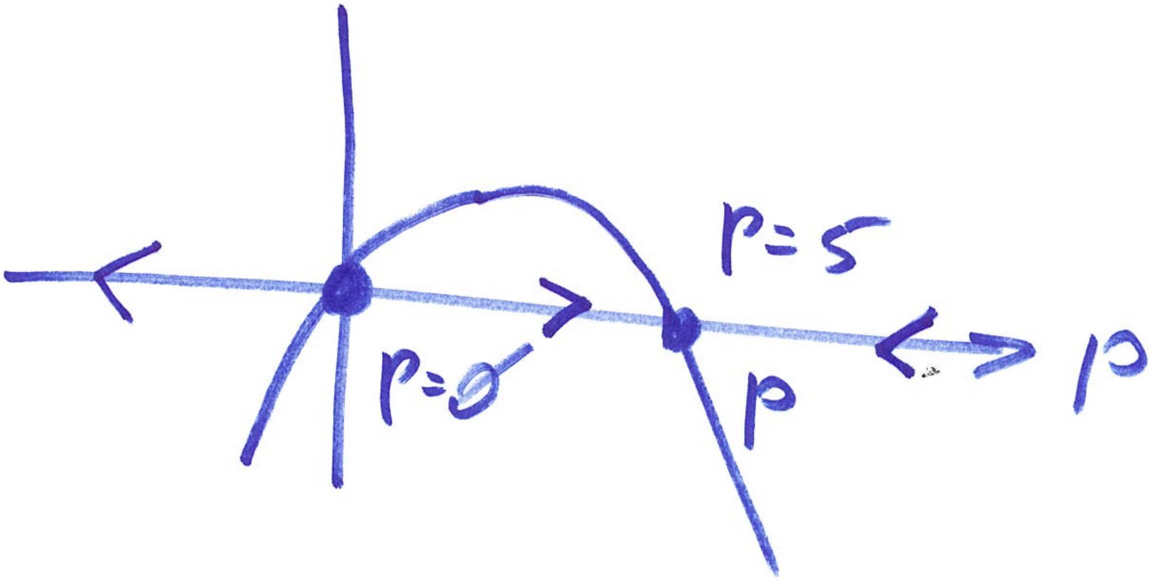
2nd Newton law

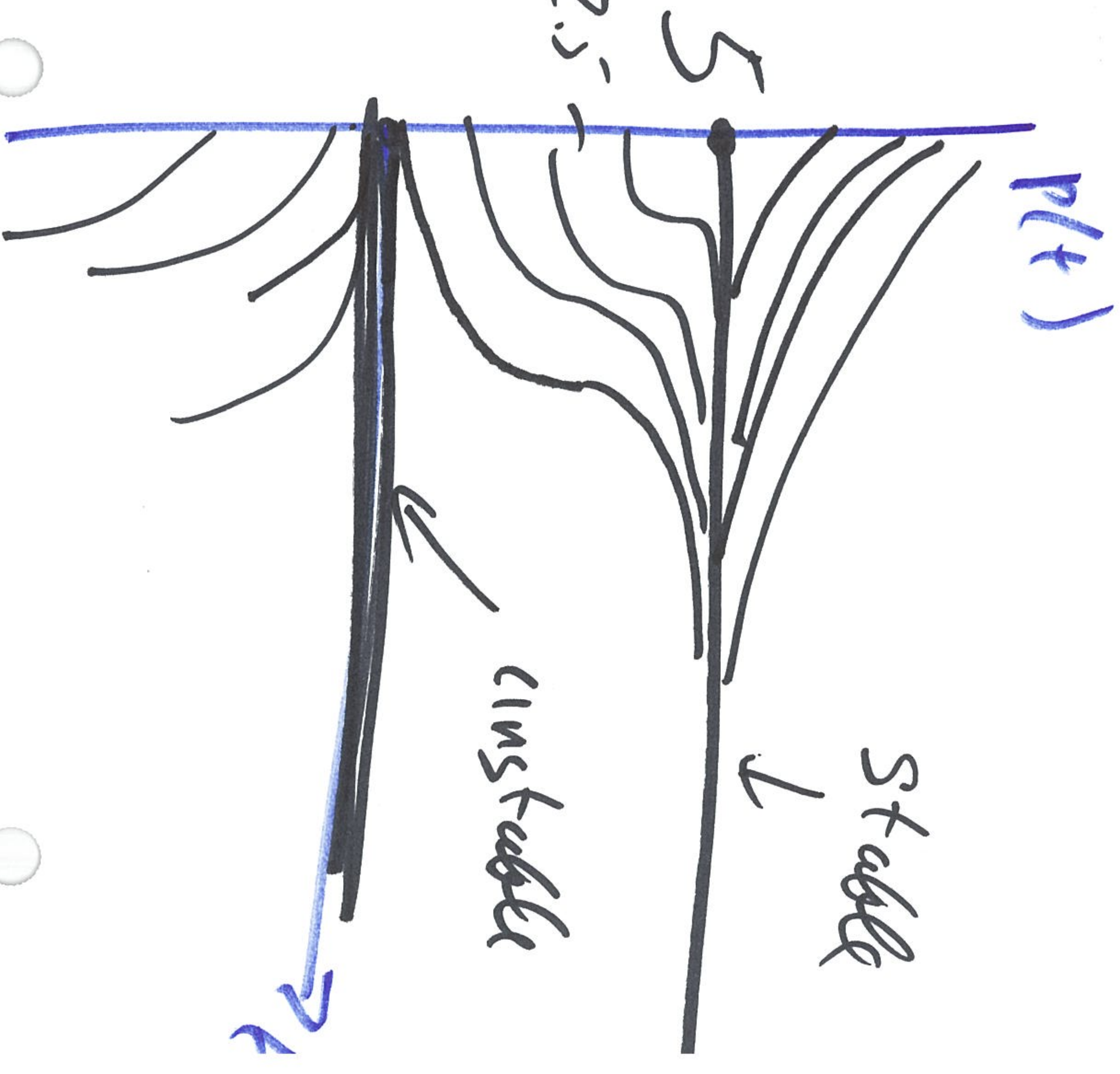
mixing.

$$\frac{d}{dt}\,\text{Smth} = \underset{\underline{\text{IN}}}{\text{Smth}} - \underset{\text{Out}}{\text{Smth}}$$

$$\frac{d}{dt} p(t) = 4(5-p)p$$

p





Q1

$$\frac{d}{dt} x(t) = f(x(t))$$

- Find x^*: $f(x^*) = 0$
- If for some x $f(x) > 0$

  x increases

- if -''- $f(x) < 0$

  $x(t)$ decreases

Sep 15

A

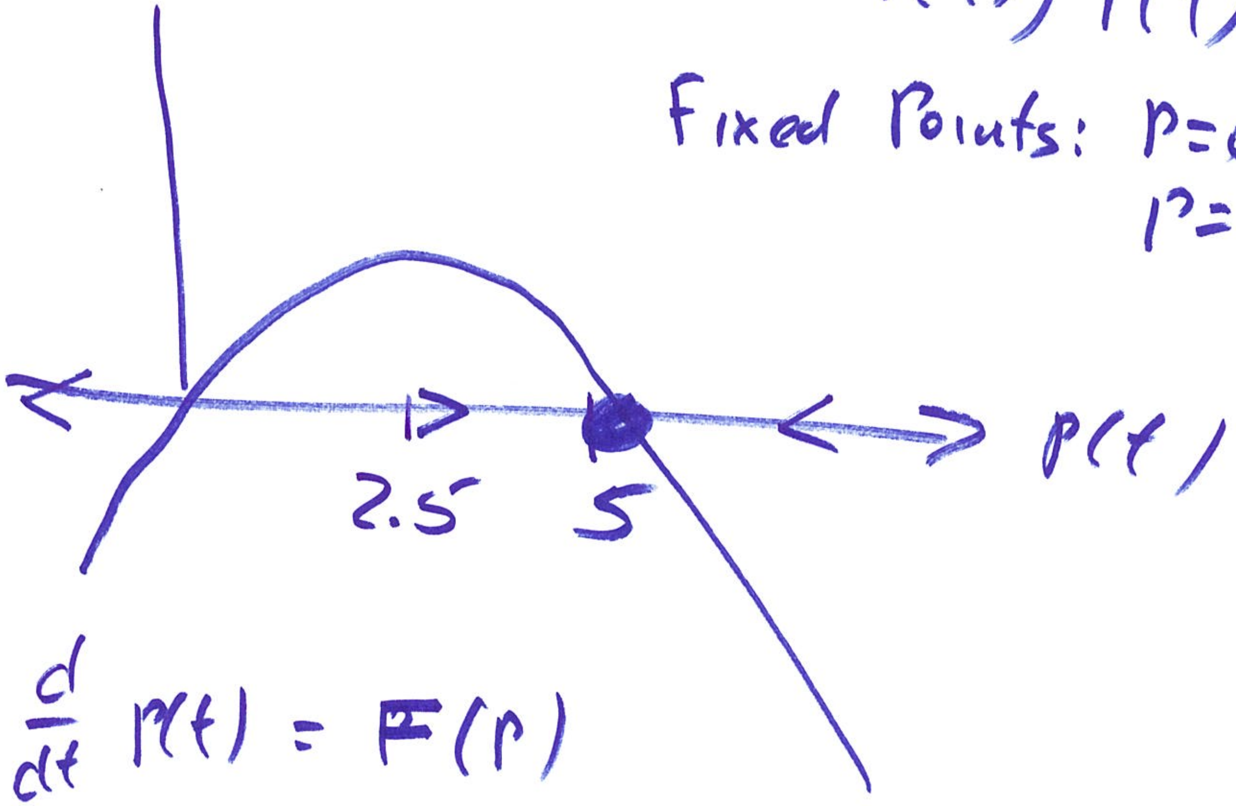
L5

Phase Plane

$$\frac{d}{dt}P(t) = 4(5 - P(t))P(t);$$

Fixed Points: $P=0$

$P=5$



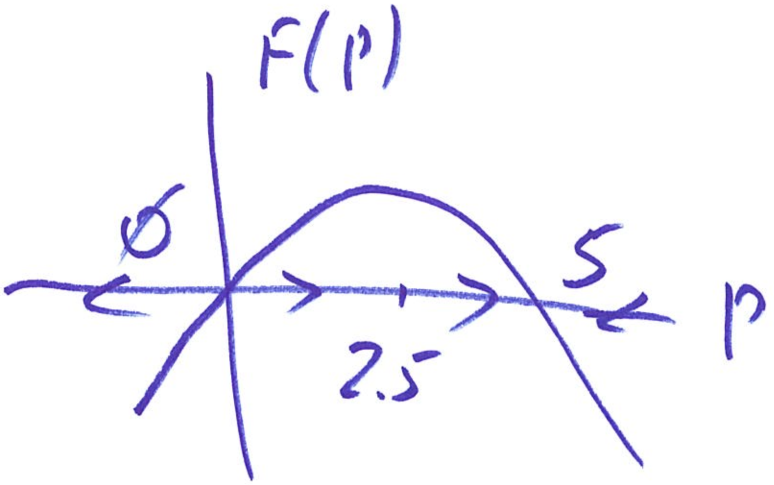
$$\frac{d}{dt}P(t) = F(P)$$

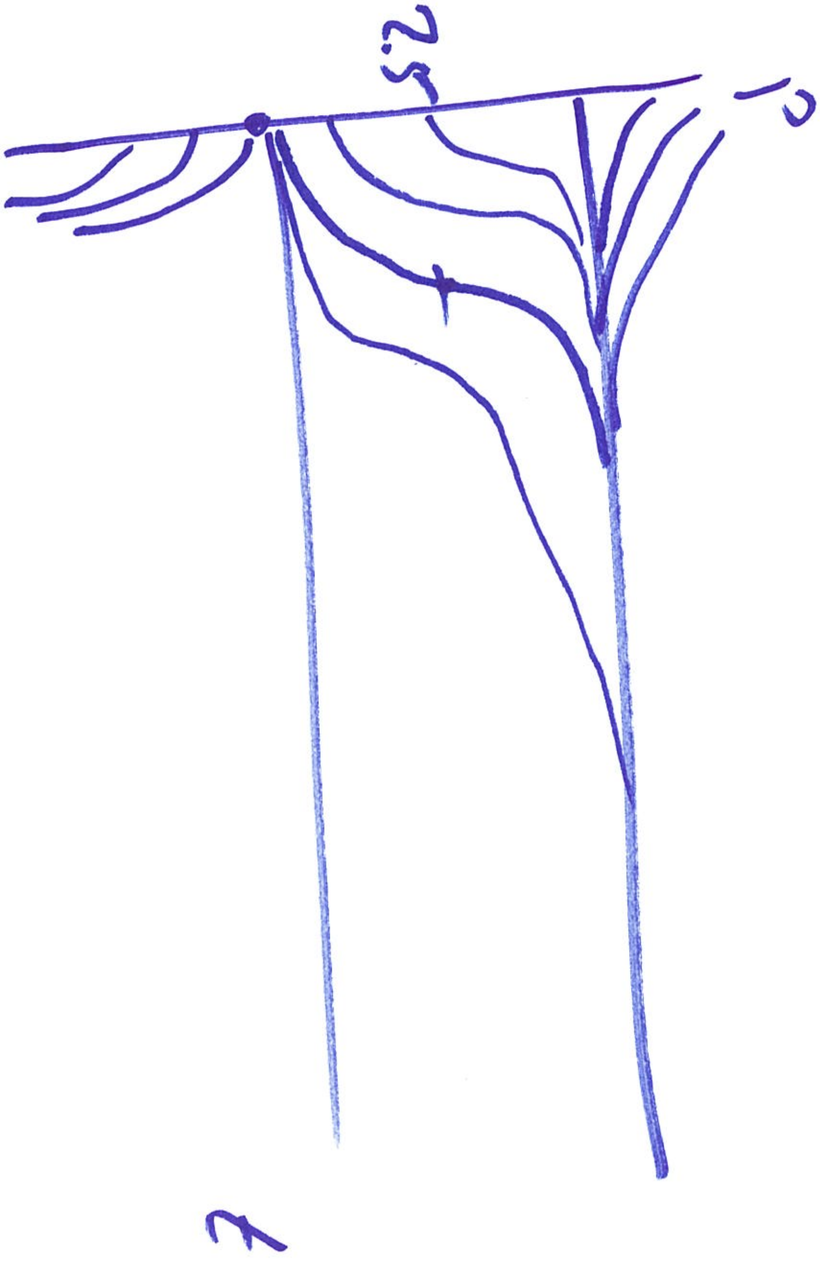
$$F(P) = 20P - 4P^2$$

$$F'(P) = 20 - 8P;$$

$$F'(P) = 0 = 20 - 8P \Rightarrow P = \frac{20}{8} = \frac{5}{2}$$

C





$\dot{x}(t) = f(x(t))$; f(x) is continious D

→ find Fixed Points: x^*

Solve $f(x^*) = 0$

→ areas of x between x^* have definite sign

If $f(x) > 0$, $x(t)$ increases
$f(x) < 0$, $x(t)$ decreases

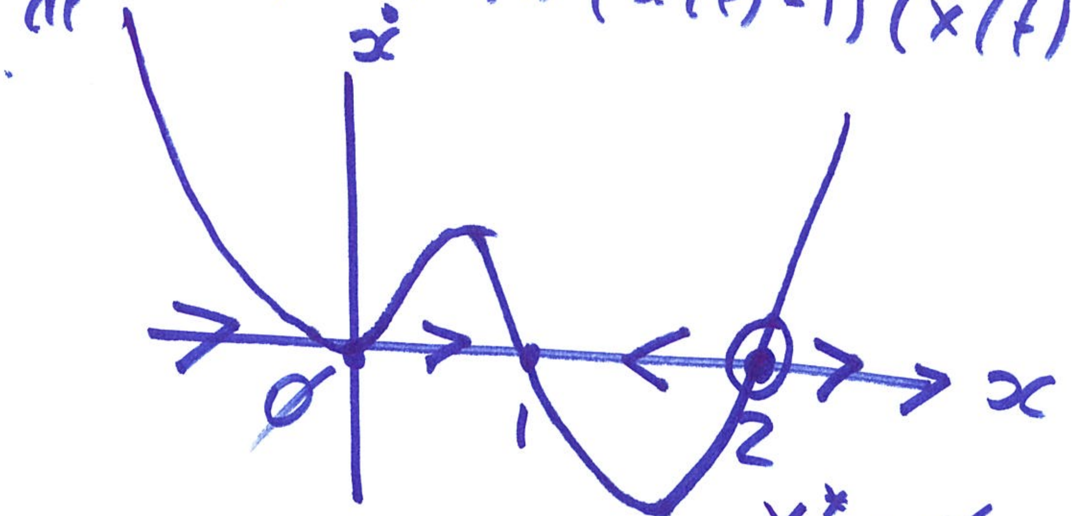
→ If $\dot{x} = f(x)$; $x(t=0) = x^*$
Then $X(t) = X^*$ for all values of t

→ stable: if
$x(t=0) = X^* + \varepsilon$, $|\varepsilon| \ll 1$
then $\cancel{\lim_{x(t)} = x^*}$; $\lim_{t\to\infty} X(t) = x^*$

→ $\lim_{t\to\infty} X(t) \neq x^*$ unstable

E

$$\frac{d}{dt} x(t) = x^2(t)(x(t)-1)(x(t)-2);$$



$x^* = 0 \Rightarrow$ mixed stability

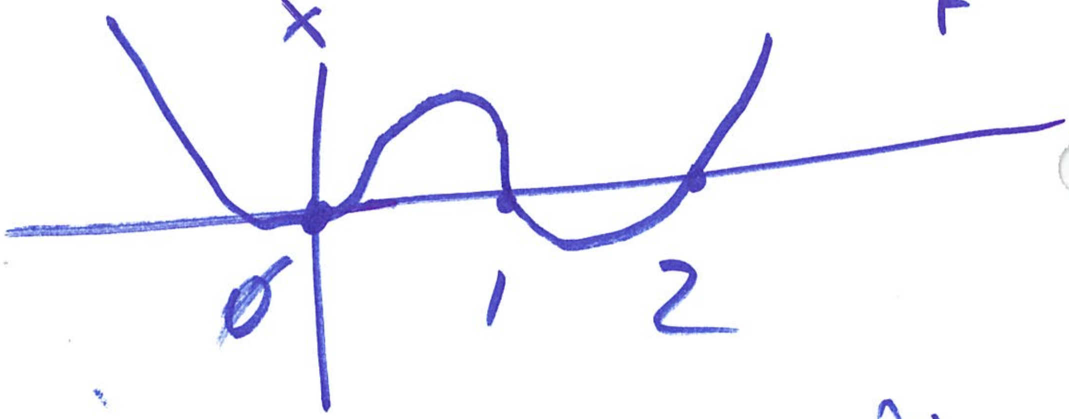
$x^* = 1$; stable

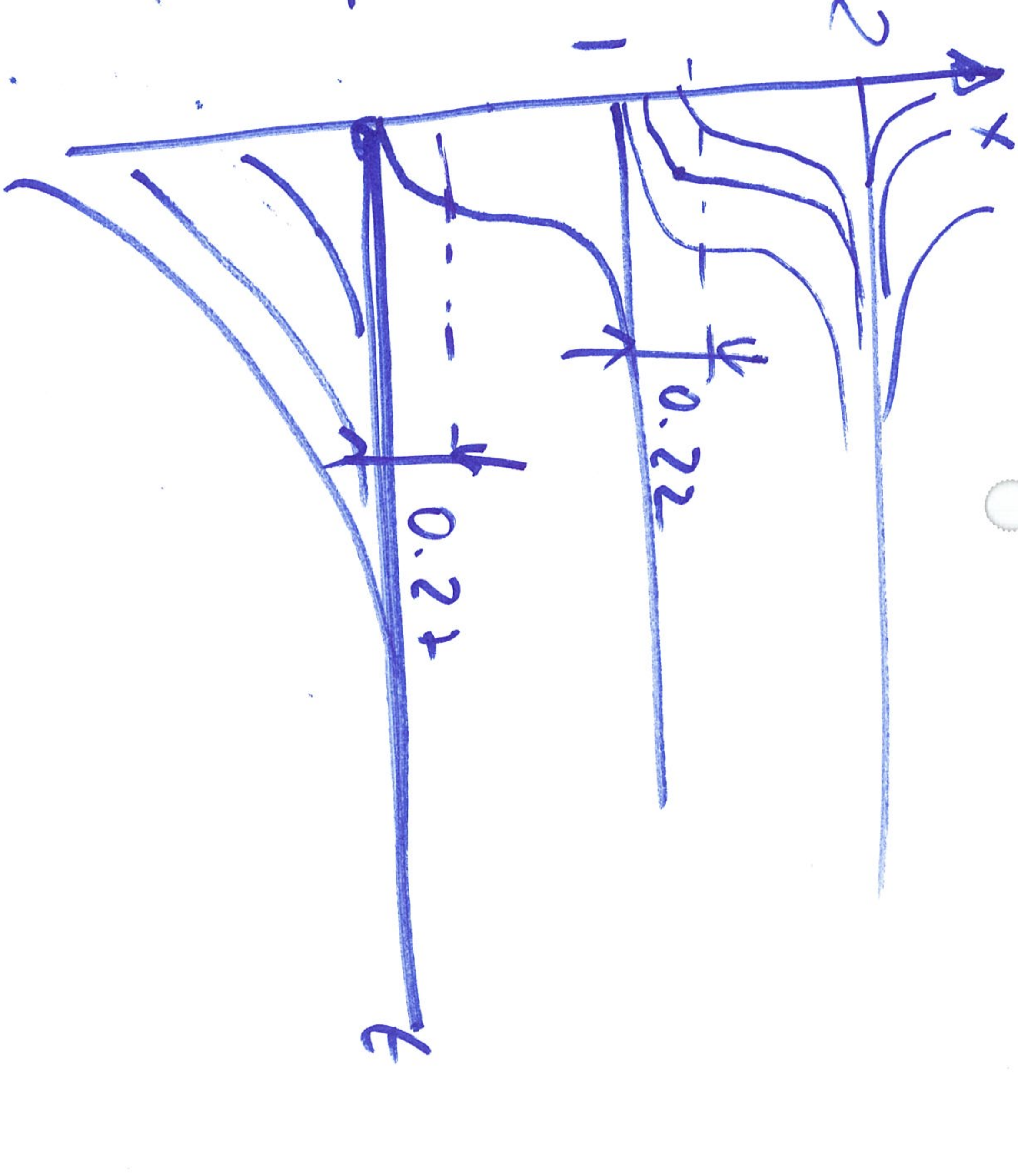
$x^* = 2$; unstable

$x^* = 0+ = 0^+$ unstable

$x^* = 0^-$. stable

F





G

$\dot{x}(t) = f(x);$

$f(x) = x^2(x-1)(x-2)$

$= x^2(x^2-3x+2) =$

$= x^4 - 3x^3 + 2x^2;$

$f'(x) = 4x^3 - 9x^2 + 4x$

$f''(x) = 12x^2 - 18x + 4 = 0$

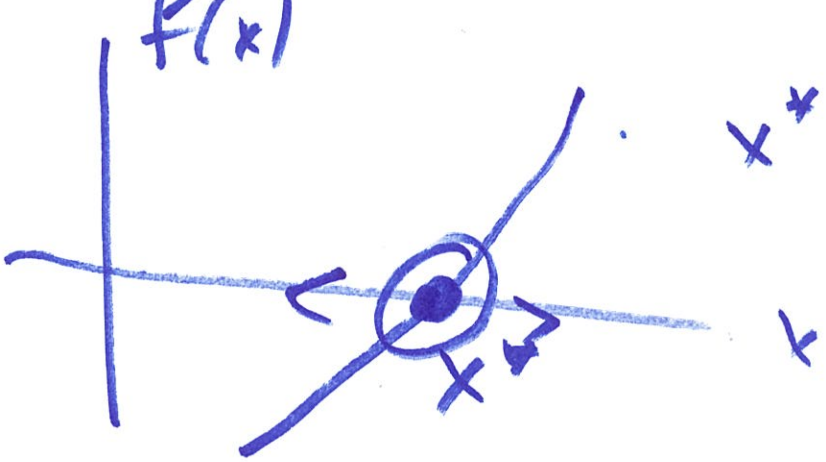
$$f''(x) = \left(x - \frac{9+\sqrt{33}}{12}\right) \cdot$$

$$\cdot \left(x - \frac{9-\sqrt{33}}{12}\right)$$

$$\approx (x - 1.22) \cdot (x - 0.27)$$

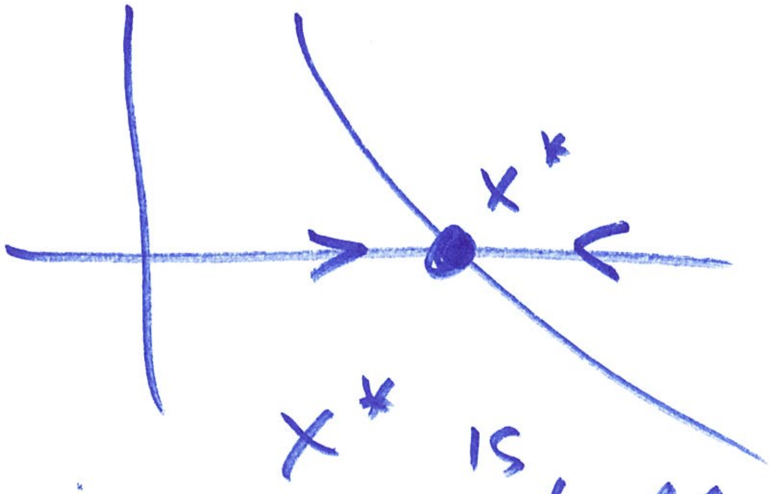
$\dot{x} = f(x)$;

- x^* s.t. $f(x^*) = 0$



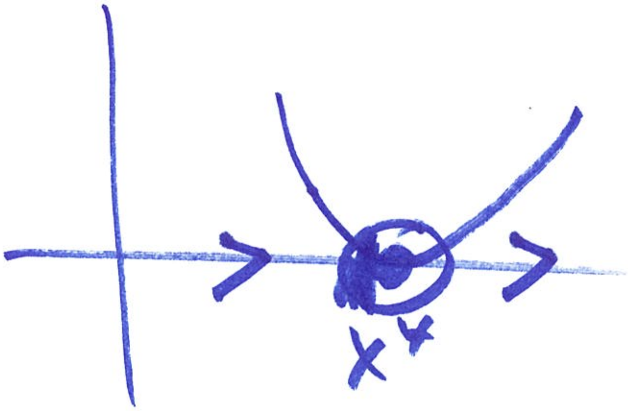
x^* is unstable

- x^*



x^* is stable

I



IF $f(x^*) = 0$

and $f(x)$ touches from above,

~~x^*~~

$x = x^* + 0$ - unstable

$x = x^* - 0$ - stable

→ classify DFQ

→ separation of variables

$$\frac{d}{dt}x(t) = f(x)g(t);$$

$$\int \frac{dx}{f(x)} = \int g(t)\,dt$$

→ integrating factor

linear first order ODE:

$$y'(x) + p(x)y(x) = g(x);\ e^{\int_0^x p(x)dx}$$

$$\frac{d}{dx}\left(y(x)e^{\int_0^x p(x)dx}\right) = g(x)\,e^{\int_0^x p(x)dx}$$

Modeling

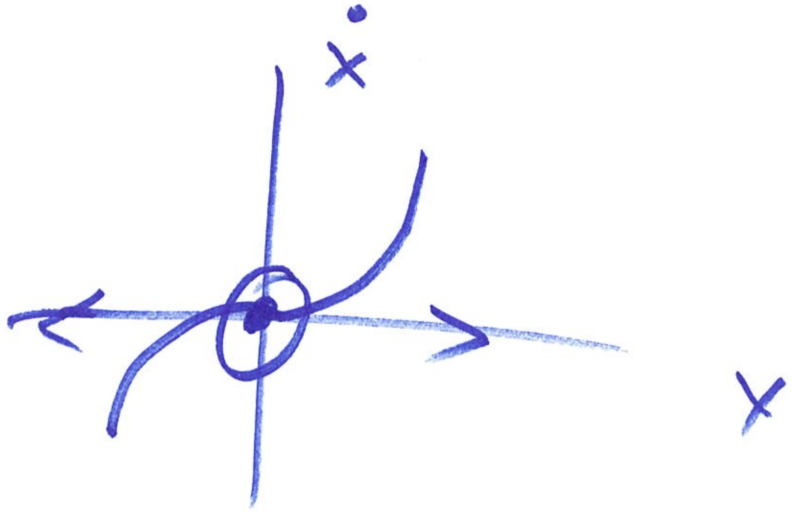
$$\frac{d}{dt} x(t) = \text{In} - \text{Out};$$

- population growth
- radioactive decay
- mixing
- 2nd Newton law
- Newton law of cooling
- Nonlinear cooling

Phase Plane

K

$\dot{x} = x^3;$



L

Linear Second order ODE

Nonelinear second order ODE: ~~$\frac{d^2}{dt^2}x(t)=f(x(t),$~~

$$\frac{d^2}{dt^2}x(t) = f\left(x(t), \frac{d}{dt}x(t), t\right)$$

Linear Second order ODE

$$y''(x) + p(x)y'(x) + \underline{g}(x)y(x) = f(x$$

No general solution;

M

General linear
2nd order ode:

$$\text{IVP}\begin{cases} y''(x) + p(x)y'(x) + g(x)\,y(x) = 0; \quad (*) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ODE} \\ y(x=0) = y_0 \qquad \text{Initial} \\ y'(x=0) = y_0' \qquad \text{conditions} \end{cases}$$

IVP - initial value problem

Let $(*)$ has two solutions $y_1(x)$ and $y_2(x)$;

Then $y(x) = C_1\, y_1(x) + C_2\, y_2(x)$ is also a sltn of $(*)$

Equation

N

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = 0$$

y_1 is a sltn

$$C_1\left(y_1''(x) + p(x)\,y_1'(x) + g(x)\,y_1(x) = 0\right)$$

$$C_2\left(y_2''(x) + p(x)\,y_2'(x) + g(x)\,y_2(x) = 0\right)$$

y_2 is a sltn

$$c_1 y_1'' + c_1 p y_1' + c_1 g y_1 \cancel{= 0} +$$

$$c_2 y_2'' + c_2 p y_2' + c_2 g y_2 = 0$$

$$\frac{d^2}{dx^2}(c_1 y_1 + c_2 y_2) + p \frac{d}{dx}(c_1 y_1 + c_2 y_2)$$

$$+ g(c_1 y_1 + c_2 y_2) = 0$$

P

Linearly independent functions:

if $y_1(x), y_2(x)$

then $y_1(x), y_2(x)$ are linearly independent if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \forall x$$

$$\Updownarrow$$

$$c_1 = c_2 = 0$$

Example: $y_1(x) = \cos x$

$y_2(x) = \sin x$

φ

Functions

$y_1(x), y_2(x) .. \; y_n(x)$

are linearly independent, if

$$c_1 y_1(x) + c_2 y_2(x) + .. + c_n y_n(x) = 0$$

$$\Updownarrow$$

$$c_1 = c_2 = \dots \; c_n = 0$$

S

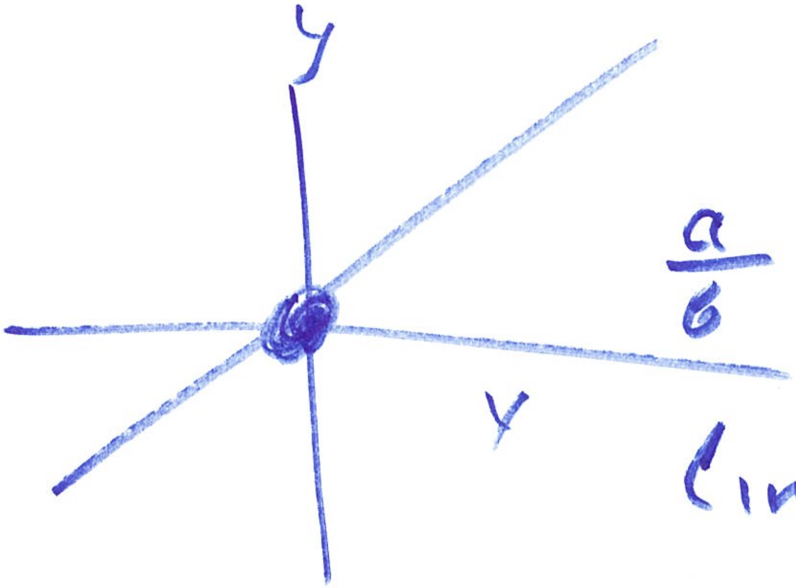
$y_1(x) = x$

$y_2(x) = 2x$ > linearly dependent

T

$$\left.\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}\right\} \quad y = -\frac{ax}{b}$$

$$y = -cx/d$$

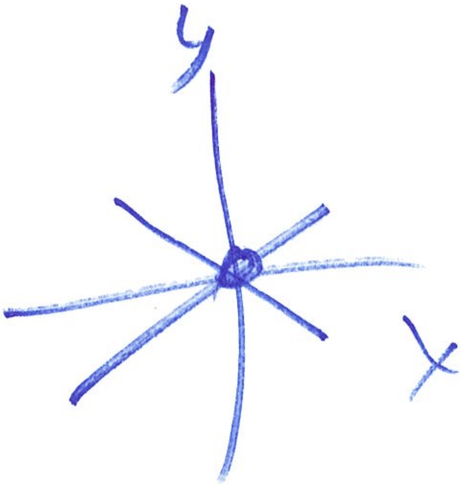


$$\frac{a}{b} = \frac{c}{d}$$

a

lines coinside

∞ many solutions



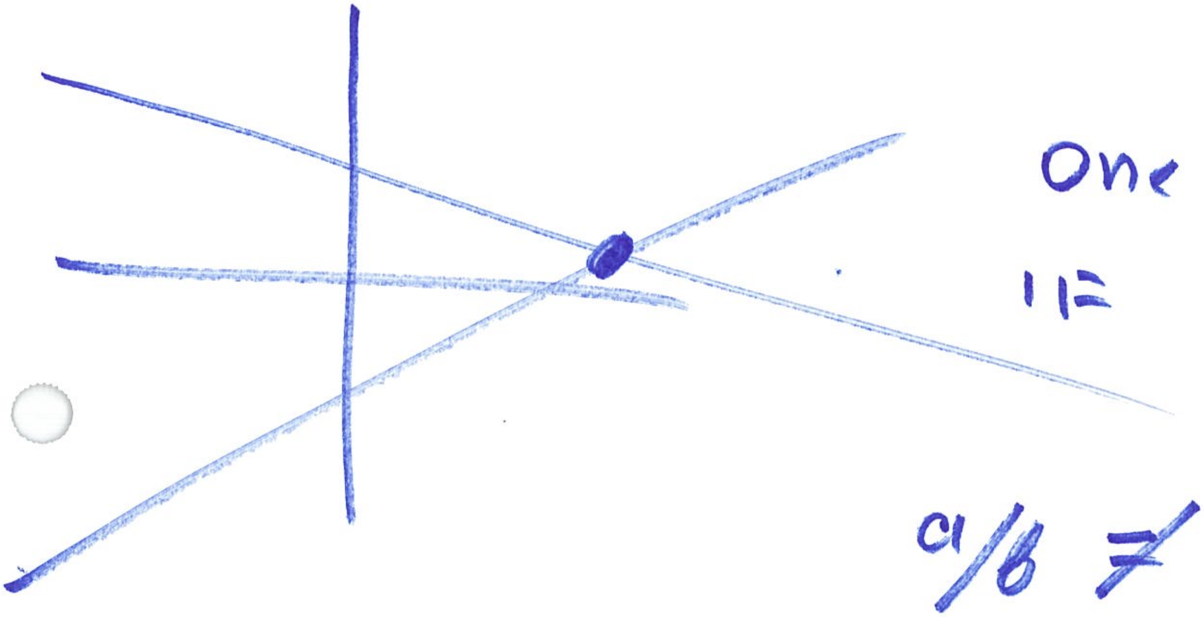
$$\frac{a}{b} \neq \frac{c}{d}$$

The only solution

is $x = y = 0$

$$\left.\begin{aligned} ax + by &= m \\ cx + dy &= e \end{aligned}\right\} \quad y(x) = \frac{m - ax}{b}$$

$$y(x) = \frac{e - cx}{d}$$



One solution

IF

$$a/b \neq \frac{c}{d}$$

IF $a/b = c/d$ then $\rightarrow$ no soln

$\rightarrow$ ∞ many

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ e \end{pmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}; \quad \underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

✓

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant

$$D = ad - bc$$

System $\underline{\underline{A}}\,\underline{x} = \underline{b}$

has unique solution

iff $\mathrm{Det}(A) \neq 0$

$$(**)\begin{cases} y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0 \quad (*) \\ y(x = x_0) = y_0 \\ y'(x = x_0) = y_0' \end{cases}$$

IVP

$y(x) = C_1 y_1(x) + C_2 y_2(x) \Rightarrow$ is a sltn to $(*)$

Find C_1, C_2 so that IVP $(**)$ is satisfied:

~~$C_1 y_1''(x_0) + C_2 y_2'(x_0) +$~~

a. $$\begin{cases} y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) = y_0 \\ y'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) = y_0' \end{cases}$$

Y

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = W$$

$$\text{Det}(W) \neq 0$$

$\boxed{A}$ L6

$$y''(x) + p(x)\, y'(x) + g(x)\, y(x) = f(x)$$

$$\Downarrow$$

homogeneous eqns:

(*) $y''(x) + p(x)\, y'(x) + g(x)\, y(x) = 0$;

$y_1(x)$, $y_2(x)$ are sltns to (*)

Then $y(x) = C_1 y_1(x) + C_2 y_2(x)$

is also a solution of (*)

y_1, y_2 are linearly independent

if $C_1 y_1(x) + C_2 y_2(x) = 0$

$$\Updownarrow$$

$C_1 = C_2 = 0$;

B

Eg (*)

$$\boxed{(*)\ y'' + py' + qy = 0;}$$

IVP

$$\boxed{\begin{aligned} y(x=0) &= y_0 \\ y'(x=0) &= y_0' \end{aligned}}$$

$$\boxed{y_1(x),\ y_2(x);}$$

two solutions to (*)

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = 0$$

$$\left.\begin{aligned} C_1 y_1(x=0) + C_2 y_2(x=0) &= y_0 \\ C_1 y_1'(x=0) + C_2 y_2'(x=0) &= y_0' \end{aligned}\right\}$$

system of eqns for $C_1 C_2$ 2 satisfy IC

$$\begin{pmatrix} y_1(x=0) & y_2(x=0) \\ y_1'(x=0) & y_2'(x=0) \end{pmatrix}$$

$$\begin{pmatrix} y_1(x=0) & y_2(x=0) \\ y_1'(x=0) & y_2'(x=0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

C

$$D\begin{pmatrix} y_1(x=0) & y_2(x=0) \\ y_1'(x=0) & y_2'(x=0) \end{pmatrix} \neq 0$$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

Wronskian 1776 - 1853

nontrivial sltns to IVP exists and are ~~uniq~~ unique iff $W(x) \neq 0$;

$w(x) \neq 0 \iff y_1, y_2$ are linearly independent.

D

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathrm{Det}(A) = ad - bc$$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

$$= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)$$

$$\begin{cases} y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$

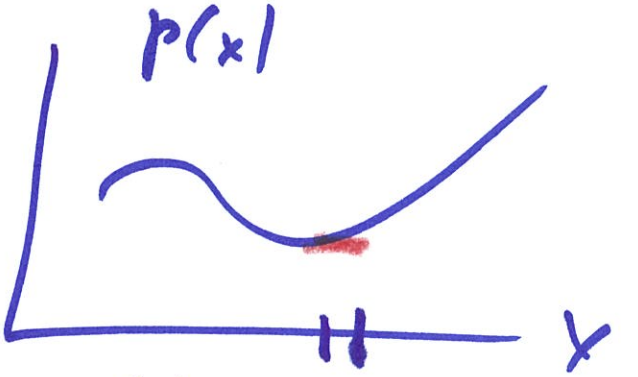
$y(x)$ is a sltn to $(*)$

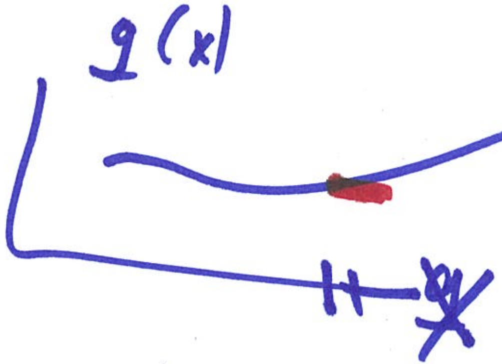
then $\tilde{y}(x) = y(x+a)$

is also a sltn

E

Solving second order linear ODE with constant coefficients.





$$a y''(x) + b y'(x) + c y(x) = 0$$

Look for solution

a, b, c are real

$$y(x) = e^{rx};$$

$$y'(x) = r e^{rx};$$

$$y''(x) = r^2 e^{rx};$$

F

$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0;$

$e^{rx} \neq 0$

$ar^2 + br + c = 0;$ characteristic equation.

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 > 4ac$, 2 real distinc roots
- $b^2 = 4ac$ 1 real "double" root
- $b^2 < 4ac$ 2 complex roots

G

$$b^2 > 4ac$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$y_1(x) = e^{r_1 x}$$

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

$$y_2(x) = e^{r_2 x}$$

$$W(x) = \mathrm{Det} \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} =$$

$$= \mathrm{Det} \begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix}$$

$$= -(r_1 - r_2) e^{(r_1 + r_2)x} \neq 0$$

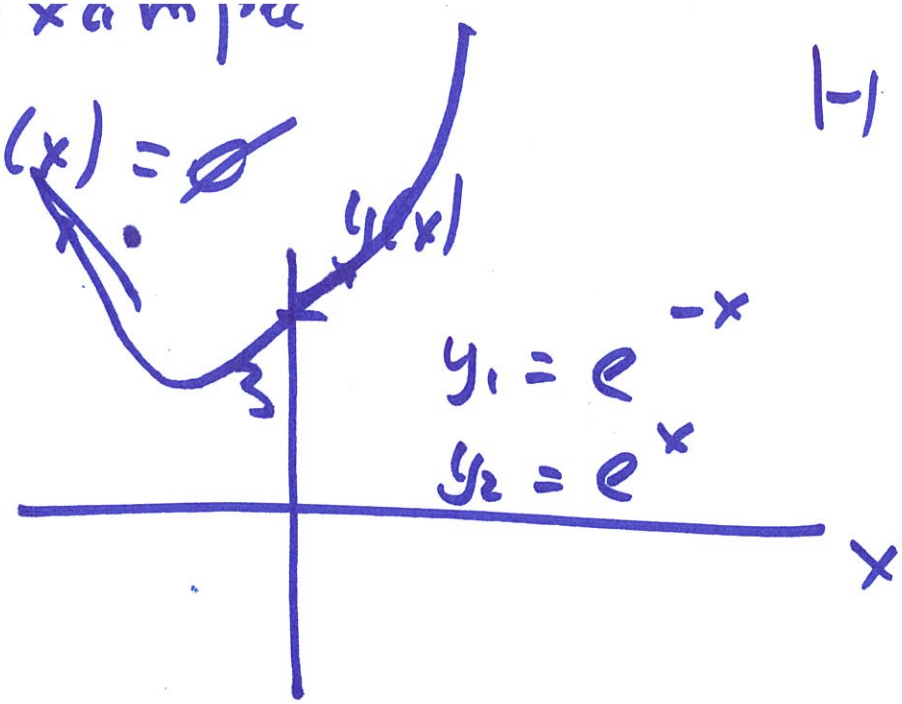
$$\Leftrightarrow \quad r_1 \neq r_2$$

Example

$y''(x) - y(x) = 0$

$y(0) = 3$

$y'(0) = 1$



$y_1 = e^{-x}$

$y_2 = e^{x}$

Soln

$y(x) = e^{rx}$; $y''(x) = r^2 e^{rx}$

$(r^2 - 1) e^{rx} = 0 \Rightarrow$

$(r-1)(r+1) = 0 \Rightarrow$ $r_1 = -1$

$r_2 = +1$

$y(x) = C_1 e^{-x} + C_2 e^{x}$

$y'(x) = -C_1 e^{-x} + C_2 e^{x}$

$y(0) = C_1 + C_2 = 3$ $\quad 2C_2 = 4$

$y'(0) = -C_1 + C_2 = 1$ $\quad C_2 = 2$

$y(x) = e^{-x} + 2e^{x}$ $\quad C_1 = 1$

I

$$W(x) = \mathrm{Det}\begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$$

$$= 1 + 1 = 2 \neq 0$$

~~ex~~ J

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!}$$

$$e^{rx} = \sum_{k=0}^{\infty} \frac{r^{k} x^{k}}{k!}$$

$$e^{r(x+\Delta)} = \sum_{k=0}^{\infty} \frac{r^{k} (x+\Delta)^{k}}{k!}$$

$$(x+\Delta)^{k} \approx x^{k} + k x^{k-1} \Delta + \ldots$$

$$\frac{d}{dx} e^{rx} = r e^{rx}$$

$$e^{r(x+\Delta)} - e^{rx} = \sum_{k=0}^{\infty} \frac{r^{k} x^{k-1} \cdot k \cdot \Delta}{k!}$$

$$k/k! = \frac{1}{(k-1)!}$$

K

$$y(x) = C_1 e^{-x} + C_2 e^{x}$$

$$\frac{d}{dx} e^{-x} = - e^{x}$$

$$\frac{d}{dx} e^{x} = e^{x}$$

$$\frac{d}{dx} y(x) = - C_1 e^{-x} + C_2 e^{x}$$

Example

L

$$y''(x)+5y'(x)+6y(x)=0$$

$$y(0)=2;\ y'(0)=3;$$

Solution

$$y(x)=e^{rx};$$

$$r^2+5r+6=0$$

$$r_{12}=\frac{-5\pm\sqrt{25-24}}{2}$$

$$=\frac{-5\pm1}{2}=-3;-2$$

$$y(x)=C_1e^{-3x}+C_2e^{-2x};$$

$$y'(x)=-3C_1e^{-3x}-2C_2e^{-2x};$$

$$y(0)=C_1+C_2=2$$

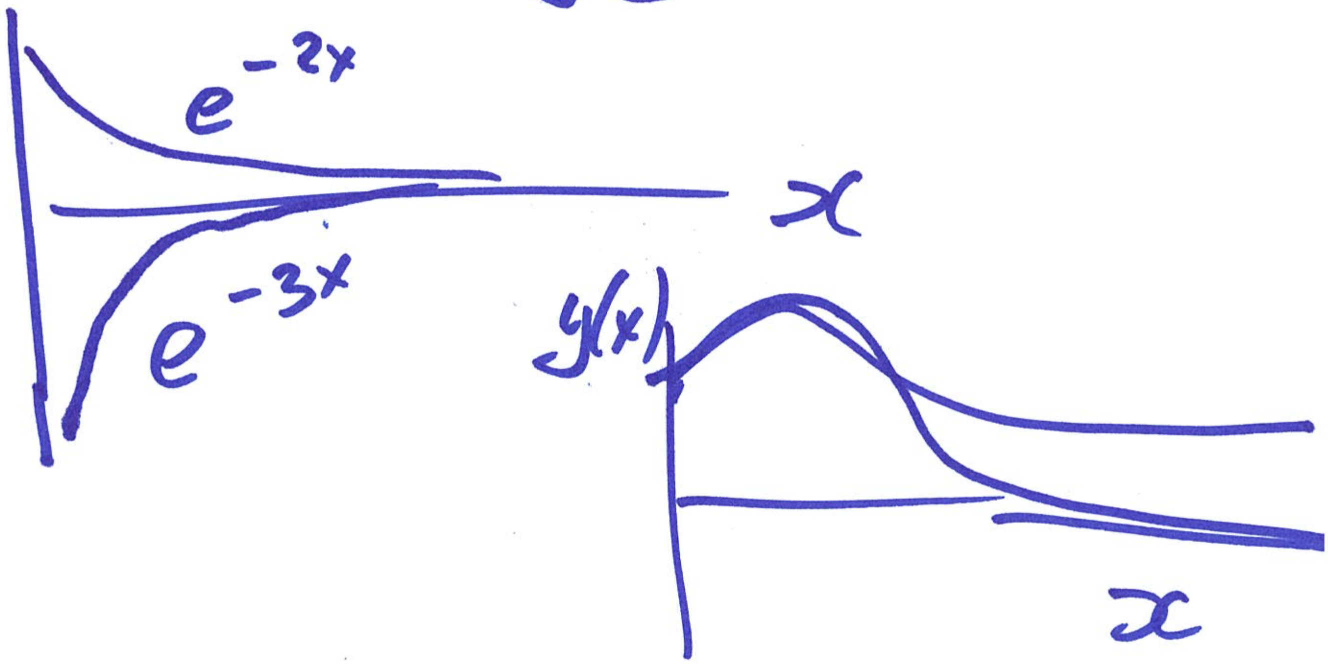
$$y'(0)=-3C_1-2C_2=3$$

M

$$2C_1 + 2C_2 = 4$$
$$-3C_1 - 2C_2 = 3$$

$$-C_1 = 7 \Rightarrow C_1 = -7$$
$$C_2 = 9$$

$$y(x) = -7e^{-3x} + 9e^{-2x}$$



N

1, 3

$(r-1)(r-3)=0$

$r^2-4r+3=0$

$$\boxed{y''(x)-4y'(x)+3y(x)=0}$$

$y(x)=C_1e^{x}+C_2e^{3x}$

$y'(x)=C_1e^{x}+3C_2e^{3x}$

$C_1=2 \quad C_2=4$

$$\boxed{\begin{aligned} y(0)&=6\\ y'(0)&=14 \end{aligned}}$$

Double root

$$(r-a)^2=0$$

$$r^2-2ar+a^2=0;$$

$$y''(x)-2ay'(x)+a^2y(x)=0$$

$$\left(\frac{d^2}{dx^2}-2a\frac{d}{dx}+a^2\right)y(x)=0$$

$$\left(\frac{d}{dx}-a\right)\boxed{\left(\frac{d}{dx}-a\right)y(x)}=0 \quad \Rightarrow z(x)$$

$$\frac{dz(x)}{dx}-az(x)=0;\ z(x)=C_1e^{ax}$$

$$\frac{dy(x)}{dx}-ay(x)=C_1e^{ax}$$

$$\frac{d}{dx}\left(y(x)e^{-ax}\right)=C_1$$

$$y(x)\, e^{-ax} = C_1 x + C_2$$

$$y(x) = C_1 x e^{ax} + C_2 e^{ax}$$

$$y_1(x) = x e^{ax}$$

$$y_2(x) = e^{ax}$$

$$W(x) \overset{D}{=} \begin{pmatrix} x e^{ax} & e^{ax} \\ e^{ax} + a x e^{ax} & a e^{ax} \end{pmatrix}$$

$$= a x e^{2ax} - e^{2ax} - a x e^{2ax}$$

$$= - e^{2ax} \neq 0$$

④

$$ay''(x) + by'(x) + cy(x) = 0$$

IF $b^2 = 4ac$

$$r = -\frac{b}{2a}$$

$$y(x) = c_1 e^r + c_2 x e^r$$

L · A

September 22

2023

=

Linear Second order ODE with constant coefficients

$$ay''(x) + by'(x) + cy(x) = 0$$

$y(x) = e^{rx}$,

$ar^2 + br + c = 0$.

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(*) $y(x) = C_1 e^{r_1 x} + C_2 \exp(r_2 x)$

$\exp(x) \equiv e^x$

- $b^2 > 4ac$, then (*)
- $b^2 = 4ac \Rightarrow y(x) = C_1 e^{rx} + C_2 x e^{r}$

$y(x) = C_1 e^{rx} + C_2 x e^{rx}$, $r = -\frac{b}{}$

C

Today: $b^2 < 4ac$

$\operatorname{Im}(a) = \operatorname{Im}(b) = \operatorname{Im}(c) = 0.$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$$

$$= \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

$r_1 = r_2^*$ - two roots are cc of each other

$r_1 = \overline{r_2}$

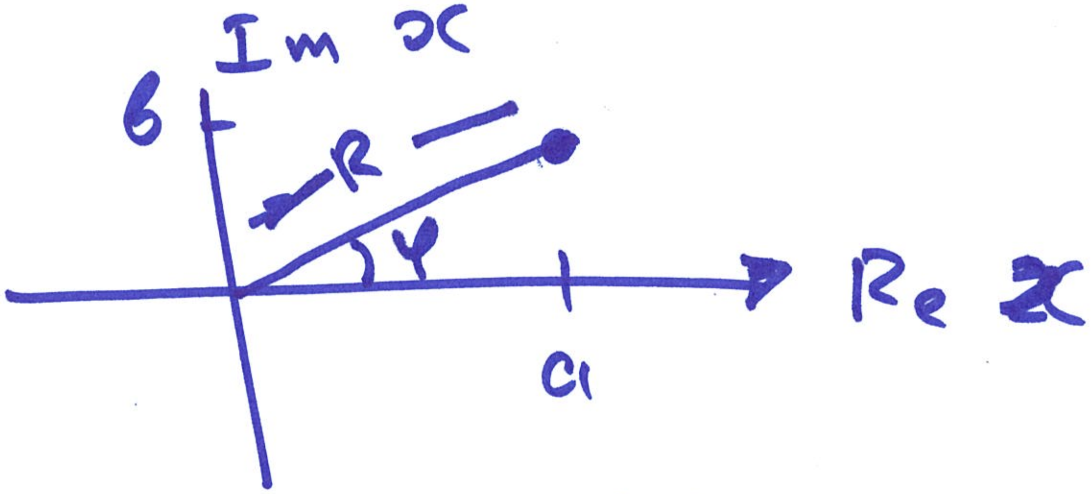
D

$\sqrt{-1} = i,$

$x = a + ib,$

a, b are real,

$a = \mathrm{Re}\, x, \ b = \mathrm{Im}\, x;$



$x = a + ib = R e^{i\varphi},$

$R = \sqrt{a^2 + b^2}$

$\tan \varphi = b/a$

$a = R \cos \varphi$

$b = R \sin \varphi$

E

Euler Equality

$$e^{ix} = \cos x + i \sin x;$$

$$e^{x} \equiv \sum_{k=0}^{\infty} \frac{x^{k}}{k!};$$

$$i^{0} = 1; \ (i)^{1} = i; \ (i)^{2} = -1;$$
$$(i)^{3} = -i; \ (i)^{4} = 1$$

$$e^{ix} = \sum_{k=0}^{\infty} \frac{i^{k} x^{k}}{k!} =$$

$$= \left(\sum_{\substack{m=0 \\ k=2m+1}}^{\infty} + \sum_{\substack{m=0 \\ k=2m}}^{\infty} \right) \frac{i^{k} x^{k}}{k!}$$

$$e^{ix} = \sum_{m=0}^{\infty} \frac{i(-1)^m x^{2m+1}}{(2m+1)!}$$ F

$\sin x$

$$+ \sum_{m=0}^{\infty} \frac{(-)^m x^{2m}}{(2m)!}$$ $\cos x$

$= \cos x + i \sin x$;

G

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\boxed{\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}}$$

$$\sin 2x = \frac{e^{2ix} - e^{-2ix}}{2i} = 2 \sin x \cos x$$

$$= \frac{(e^{ix})^2 - (e^{-ix})^2}{2i} =$$

$$= \underbrace{\boxed{\frac{(e^{ix} + e^{-ix})}{2}}}_{\cos x} \cdot \overbrace{\boxed{\frac{(e^{ix} - e^{-ix})}{2i}}}^{\sin x} 2$$

$$a\,y''(x) + b\,y'(x) + c\,y(x) = 0$$

$$b^2 < 4ac$$

$$r = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

$$= \underbrace{-\frac{b}{2a}}_{\gamma} \pm i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\omega}$$

$$= \gamma \pm i\omega; \quad \gamma = -\frac{b}{2a}$$

$$\omega = \frac{\sqrt{4ac - b^2}}{2a}$$

I

$$ay''(x) + by'(x) + y(x) = 0$$

$$r_{1,2} = \gamma \pm i\omega;$$

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

1st representation

$$y(x) = C_1 e^{\gamma x + i\omega x} + C_2 e^{\gamma x - i\omega x}$$

$$= e^{\gamma x}\left[C_1 e^{i\omega x} + C_2 e^{-i\omega x}\right]$$

$$= e^{\gamma x}\left(C_1 \cos\omega x + i C_1 \sin\omega x + C_2 \cos\omega x - i C_2 \sin\omega x\right)$$

$$= e^{\gamma x}\left[\underbrace{(C_1 + C_2)}_{D_1} \cos\omega x + \underbrace{i(C_1 - C_2)}_{D_2} \sin\omega x\right]$$

$$C_1 + C_2 \equiv D_1$$

$$iC_1 - iC_2 \equiv D_2$$

$$\boxed{y(x) = D_1 e^{\gamma x} \cos\omega x + D_2 e^{\gamma x} \sin\omega x}$$

II nd representation

$$C_1 = A + iB$$

$$C_2 = A - iB$$

$$C_1 + C_2 = 2A$$

$$iC_1 - iC_2 = iA - B - iA$$

$$-B = -2B$$

$$y(x) = C_1 e^{\gamma x + i\omega x} + C_2 e^{\gamma x - i\omega x}$$

2nd $= D_1 e^{\gamma x} \cos\omega x + D_2 e^{\gamma x} \sin\omega x$

3d $= R e^{\gamma x} \cos(\omega x + \varphi)$

4th $= R e^{\gamma x} \sin(\omega x + \psi)$

$$y(x) = D_1 e^{\gamma x} \cos\omega x \in ay'' + by' + cy =$$

$$x = \tilde{x} + x_0$$

~~$y(\tilde{x}) = D_1 e^{\gamma x}$~~

$$y(x) = \boxed{D_1 e^{\gamma \tilde{x}}} e^{\gamma x_0} \cos(\omega(\tilde{x} + x_0))$$

$$= E_1 e^{\gamma \tilde{x}} \cos(\omega \tilde{x} + \varphi) = y(\tilde{x})$$

~~$E_1 = D_1 e^{\gamma x_0}$~~ $\Rightarrow E_1 = D_1 e^{\gamma x_0}$

$\varphi = \omega x_0$

$C_1 \; C_2$

$D_1 \; D_2$

$R \; \varphi$

$R \; \psi$

L

$$\cos(a+b) = \cos(a)\cos(b) - \sin a \sin b$$

$$\cos(a+b) = \frac{e^{ia+ib} + e^{-ia-ib}}{2}$$

$$(\overset{\sin}{\cos} a)(\overset{\sin}{\cos} b) = \frac{(e^{ia} + e^{-ia})(e^{ib} + e^{-ib})}{}$$

$$= \frac{1}{4}\left(e^{i(a+b)} + e^{-i(a+b)} + e^{i(a-b)}\right.$$

~~$\sin a \sin b = \frac{1}{4}\left(e^{ia} + e^{i(b-a)}\right)$~~

$$\cos a \cos b = \frac{1}{4}\left(e^{i(a+b)} + e^{-i(a+b)} \underline{+ e^{i(a-b)} + e^{i(b-a)}}\right)$$

$$-\sin a \sin b = +\frac{1}{4}\left(e^{ia} - e^{-ia}\right)\left(e^{ib} - e^{-ib}\right) =$$

$$= +\frac{1}{4}\left(e^{i(a+b)} + e^{-i(a+b)} \underline{- e^{i(a-b)} - e^{i(b-a)}}\right)$$

$$\cos a \cdot \cos b - \sin a \sin b =$$

$$= \frac{1}{2}\left(e^{i(a+b)} + e^{-i(a+b)}\right) = \cos(a+b)$$

N

$$y(x) = e^{\gamma x} \cdot R \cdot \cos(\omega x + \varphi) =$$

$$= R e^{\gamma x} \cos(\omega x) \cos\varphi - R e^{\gamma x} \sin(\omega x) \sin\varphi$$

$$D_1 = R\cos\varphi; \quad D_2 = - R\sin\varphi$$

$$y(x) = D_1 e^{\gamma x} \cos \omega x + D_2 e^{\gamma x} \sin \omega x$$

$$y''(x) - 2y'(x) + 2y(x) = 0;$$ 0

Example

$$r^2 - 2r + 2 = 0;$$

$$r_{12} = \frac{2 \pm \sqrt{4-8}}{2} =$$

$$= 1 \pm \cancel{2}i = 1 \pm i$$

$$y(x) = C_1 e^{x+ix} + C_2 e^{x-ix}$$

$$\left.\begin{aligned} y(x) &= D_1 e^x \cos x + D_2 e^x \sin x \\ y(x) &= R e^x \cos(x+\varphi) \\ y(x) &= R e^x \sin(x+\psi) \end{aligned}\right]$$

Example

p

$$y''(x) + y(x) = 0$$

$$y(x) = e^{rx}; \quad r^2 + 1 = 0$$

$$r^2 = -1; \quad r = \pm i$$

$$y(x) = A \cos x + B \sin x$$

$$y(x) = R \cos(x + \varphi)$$

$$y(x) = R \sin(x - \varphi)$$

Example

$$\boxed{y(x) = e^{x}\cos 5x - e^{x}\sin 5x}$$

$$y''(x) - 2y'(x) + 26y(x) = 0$$

$$y(0) = 1 \; ; \; y'(0) = -4 \; ;$$

Solution

$$y(x) = e^{t}\sqrt{2}\sin\left(\frac{\pi}{4} - 5\right.$$

$$r^2 - 2r + 26 = 0$$

$$r_{12} = \frac{2 \pm \sqrt{4 - 104}}{2} =$$

$$= \frac{2 \pm 10i}{2} = 1 \pm 5i \; ;$$

$$y(x) = C_1 e^{x}\cos(5x) + C_2 e^{x}\sin(5x)$$

$$y'(x) = C_1 e^{x}\cos(5x) + C_2 e^{x}\sin(5x)$$
$$- 5C_1 e^{x}\sin(5x) + 5C_2 e^{x}\cos(5x)$$

$$y(0) = C_1 \cancel{+ C_2} = 1 \qquad C_1 = 1$$

$$y'(0) = C_1 + 5C_2 = -4 \qquad C_2 = -1$$

S

$$\frac{d^4}{dx^4} y(x) - y(x) = 0$$

$$y''''(x) - y(x) = 0$$

$$y(x) = e^{rx} ; \quad y''''(x) = r^4 e^{rx}$$

$$r^4 - 1 = 0$$

$$r_1 = 1 ; \quad r_2 = -1 ;$$

$$r_3 = i ; \quad r_4 = -i$$

$$y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin(x)$$

$$y''''(x) = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + + C_4 \sin x$$

A

$\mathcal{L}$

Name on cribsheets

First order ODE → modeling

26 Sep 2023

seperation of variables

$$\frac{dy(x)}{dx} = F(x)G(y(x))$$

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Integrating Factor

$$y'(x) + p(x)y(x) = g(x)$$

$$\frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = g(x)e^{\int p(x)dx}$$

$$\frac{d}{dt}X(t) = In - Ou$$

Modeling
- → population
- → radioactive
- → interest rate
- → mixing
- → cooling

26 Sep 2023

B

$$\frac{d}{dx}\left(y(x)\left(e^x + \ln(x)\right)\right) = 1$$

$$y'(x)\left(\underline{e}^x + \underline{\ln x}\right) + y(x)\left(e^x + 1/x\right) = 1$$

$$y'(x) + \frac{\cancel{x}\left(e^x \cdot x + 1\right)}{\cancel{x}\left(e^x + \ln x\right)} = \frac{}{e^x + \ln x}$$

C

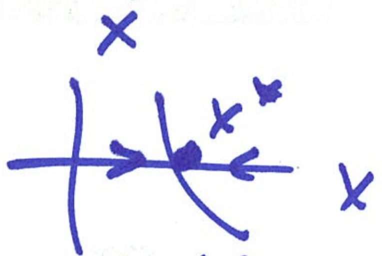
$$\frac{d}{dt} x(t) = f(x(t));$$

① Find x^* s.t. $f(x^*) = 0$

ⓐ fixed points

ⓑ stability

unstable



stable

② if $f(x) > 0$, then $x(t) \nearrow$

$f(x) < 0$, then $x(t) \searrow$

③

D

Linear Second Order

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x) = f(x);$$

homogeneous

$f(x) = 0.$

$$y(x) = C_1\,y_1(x) + C_2\,y_2(x)$$

where y_1, y_2 are solutions of $y'' + py' + gy = 0;$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

$W(x) \neq 0 \iff y_1, y_2$ are linearly independent

E

$$a y'' + b y' + c y = 0$$

$i\omega$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

γ

→ real and different r_1, r_2

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

→ real and same : $b^2 = 4ac$

$$y(x) = C_1 e^{rx} + C_2 x e^{rx}$$

→ $b^2 < 4ac$, $\gamma = -\frac{b}{2a} = -\frac{b}{2a}$

$$\omega = \frac{1}{2a}\sqrt{4ac - b^2}$$

F

①

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

② $$y(x) = D_1 e^{\gamma x} \cos \omega x + D_2 e^{\gamma x} \sin \omega x$$

③ $$y(x) = R e^{\gamma x} \cos(\omega t - \varphi)$$

④ $$y(x) = R e^{\gamma x} \sin(\omega t - \psi)$$

G

$$y(x) = D_1 e^{\gamma x} \cos \omega x + D_2 e^{\gamma x} \sin \omega x$$

$$y(x) = R e^{\gamma x} \cos(\omega \cancel{t} - \varphi) =$$

$$= R e^{\gamma x} (\cos \omega \cancel{t} \cos \varphi + + \sin \omega t \sin \varphi)$$

$$=$$

$$= \boxed{R} e^{\gamma x} \cos \omega \cancel{t} \boxed{\cos \varphi} = D_1$$

$$+ \boxed{R} e^{\gamma x} \sin \omega \cancel{t} \boxed{\sin \varphi} = D_2$$

H

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

I

$y(x) = e^{x} \cos(2x + 2)$

$y'(x) = e^{x} \cos(2x + 2)$

$-2e^{x} \sin(2x + 2)$

$y(0) = \cos 2$ $\qquad 1 + 2i$

$y'(0) = \cos 2 - \underline{2} \sin 2$

$1 + 2i$ $\qquad$ * $\qquad B = -2$

$1 - 2i$ $\qquad a = 1$

$\boxed{y'' - 2y' + 2y = 0}$ $\qquad B^{2} - 4ac = -4$

$4 - 4c = -4$

$c = 2$

J

$$y'' - 2y' + 2y = 0$$

$$y(0) = \cos 2$$

$$y'(0) = \cos 2 - \sin 2$$

$$y(x) = e^{x}\cos(x+2)$$

$$r^2 - 2r + 2 = 0$$

$$r_{12} = \frac{2 \pm \sqrt{4-8}}{2} =$$

$$= 1 \pm i$$

$$y(x) = R e^{x} \cos(x + \varphi)$$

$$y'(x) = Re^{x}\cos(x+\varphi) - Re^{x}\sin(x+\varphi$$

$$y(0) = R\cos\varphi = \cos 2$$

$$y'(0) = R\cos\varphi - R\sin\varphi =$$

$$= \cos 2 - \sin 2$$

$$R = 1,\ \varphi = 2$$

K

INHOMOGENEOUS second order linear equations.

$$y''(x) + p(x)\,y'(x) + g(x)\,y(x)$$

=

$z(x) = y_1(x) - y_2(x) = C_1 z_1(x) + C_2 z_2(x)$ M

$y_1(x) = y_2(x) + C_1 z_1(x) + C_2 z_2(x)$

y_1, y_2 are sltns to $y'' + py' + qy = f(x)$

z_1, z_2 -"- "- to $z'' + pz' + qz = 0$

General solution of linear inhomogeneous ODE is a sum of a particular solution of inhomogeneous ODE and a general solution of corresponding homogeneous ODE.

N

$$y_1(x) = C_1 z_1(x)$$

$$y_2(x) = C_2 z_2(x)$$

0

Example

$$\boxed{y(x) = -\frac{e^{2x}}{2} + C_1 e^{-x} + C_2 e^{4x}}$$

$$y''(x) - 3y'(x) - 4y(x) = 3e^{2x}$$

① KILL RHS

$$y_0'' - 3y_0' - 4y_0 = 0$$

$y_0 \equiv$ homogeneous equation

$$r^2 - 3r - 4 = 0$$

$$r_{12} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$$

$$r_1 = 4 \qquad r_2 = -1$$

$$y_0(x) = C_1 e^{-x} + C_2 e^{4x}$$

(II) inhomogeneous eqn:

$$y(x) = A e^{2x}; \quad y' = 2Ae^{2x}$$

$$4A - 6A - 4A = 3 \qquad y'' = 4Ae^{2x}$$

$$A = -1/2$$

p

$$y''(x) - 3y'(x) - 4y(x) = 2\sin x$$

$$y_p(x) = A\sin x$$

$$y_p'(x) = A\cos x$$

$$y_p''(x) = -A\sin x$$

$$-A\sin x - 3A\cos x - 4A\sin x = 2\sin x$$

$$-3A = (2 + 5A)\tan x$$

Q

$y_p = B\cos x$

$y_p' = -B\sin x$

$y_p'' = -B\cos x$

$$-B\cos x + 3B\sin x - 4B\cos x = 2\sin x$$

A

October 2

2023

DF_Q

B

$$y''(x) - 3y'(x) - 4y(x) = RHS(x)$$

Homogeneous : $RHS(x) = 0;$

Now inhomogeneous

$$RHS(x) = \underbrace{3e^{2x};}_{}$$

$$y(x) = \underset{\substack{\uparrow \\ \text{general} \\ \text{solution} \\ \text{of} \\ \text{homogeneous} \\ \text{equation}}}{y_0(x)} + \underset{\substack{\uparrow \\ \text{a particular} \\ \text{-"- -"-} \\ \text{inhomogeneous} \\ \text{eqn.}}}{y_p(x)}$$

$$y_p(x) = \underbrace{-\frac{1}{2}e^{2x}}_{}$$

(-1, 4)

Ⓒ $y''(x) - 3y'(x) - 4y(x) = 2\sin x;$

$y_p(x) = A \cdot \sin x;$ ✗

$y_p(x) = B \cdot \cos x;$ ✗

$$\begin{cases} y_p(x) = A \cdot \sin x + B\cos x; \\ y_p'(x) = A\cos x - B\sin x \\ y_p''(x) = -A\sin x - B\cos x; \end{cases}$$

$$\left.\begin{aligned} &-A\,\underline{\sin x} - \underline{\underline{B\cos x}} \\ &-\underline{\underline{3A\cos x}} + \underline{3B\sin x} \\ &-\underline{4A\sin x} - \underline{\underline{4B\cos x}} \end{aligned}\right\} \cdot$$

$$= \underline{2\sin x}$$

$$(\sin x)\cdot\left(\overbrace{-A + 3B - 4A - 2}^{=0}\right) +$$

$$(\cos x)\cdot\left(\underbrace{-B - 3A - 4B}_{=0}\right) = 0$$

D

$$\left.\begin{array}{l} -5A + 3B = 2 \\ -3A - 5B = 0 \end{array}\right\}$$

$$3A = -5B \; ; \; A = -\frac{5}{3}B$$

$$+\frac{25B}{3} + 3B = 2$$

$$\frac{25B}{3} + \frac{9B}{3} = 2$$

$$\frac{34}{3}B = 2; \; B = 2 \cdot \frac{3}{34} = \frac{3}{17}$$

$$A = -\frac{5}{\cancel{3}} \cdot \frac{\cancel{3}}{17} =$$ [illegible]

$$= -\frac{5}{17};$$

~~$y_g(x) = C_1 e^{-x} +$~~

$$y_G(x) = C_1 e^{-x} + C_2 e^{4x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x;$$

E

EXAMPLE

$$y'' - 3y' - 4y = 4x^2;$$

$$y(x) = A \cdot x^2 + Bx + C$$

$$y'(x) = 2A \cdot x + B$$

$$y''(x) = 2A$$

~~$2A\cancel{x^2} + Bx + C - 3Ax - 3B - 4Ax^2 - 4Bx - 4C = 4x^2$~~

$$2A - \underline{6Ax} - 3B - \underline{\underline{4Ax^2}} - \underline{Bx \cdot 4} - 4C = \underline{\underline{x^2 \cdot 4}}$$

~~x~~

$$x^2\overbrace{(-4A-4)} + x\overbrace{(-6A-4B)} + 2A - 3B - 4C = 0$$

F

$$\left.\begin{aligned} -4A &= 4 \\ -6A - 4B &= 0 \\ 2A - 3B - 4C &= 0 \end{aligned}\right\}$$

$$A = -1; \quad 6 - 4B = 0$$

$$B = \frac{6}{4} = \frac{3}{2};$$

$$-2 - \frac{9}{2} - 4C = 0$$

$$-\frac{13}{2} = 4C \Rightarrow C = -\frac{13}{8}$$

$$\left(y_p(x) = -x^2 + \frac{3}{2}x - \frac{13}{8}\right) -4$$

$$\left(y_p'(x) = -2x + \frac{3}{2}\right)(-3)$$

$$\left(y''_p(x) = -2\right) 1$$

$$0 = 0$$

I

→ $RHS(x) = A \cdot e^{Bx} \Rightarrow y_r = Ce^{Bx}$

→ $RHS(x) = A \sin x$

or

$RHS(x) = B \cos x$

$y_r(x) = C \sin x + B \cos x$

→ $RHS(x) = \sum_{k=0}^{n} A_k x^k$

$y_r(x) = \sum_{k=0}^{n} B_k x^2$

① $RHS(x) = e^{Ax} \cdot \sum_{k=0}^{n} C_k \cdot x^k (D \sin Ex + F \cos Ex)$

$y_p(x) = x^s e^{Ax} \sum_{k=0}^{n} H_k x^k (I \sin Ex + J \cos Ex)$

$s = 0, 1, 2$ so that no term
of particular solution
solves homogeneous
equation

K

$$y''(x) - 3y'(x) - 4y(x) =$$

$$= -8e^{x}\cos 2x;$$

$$y_p(x) = e^{x}(A\cos 2x + B\sin 2x)$$

$$y_p'(x) = e^{x}(A\cos 2x + B\sin 2x)$$

$$+ e^{x}(-2A\sin 2x + 2B\cos 2x)$$

$$= e^{x}((A+2B)\cos 2x + (-2A+B)\sin 2x)$$

$$y_p''(x) = e^{x}\Big[(A+2B)\cos 2x + (-2A+B) \cdot \sin 2x$$

$$-2(A+2B)\sin 2x + 2(-2A+B) \cdot \cos 2x\Big]$$

$$= e^{x}\Big[(A+2B-4A+2B)\cos 2x$$

$$+ (-2A+B-2A-4B)\sin 2x\Big]$$

y_p

L

$$\Big(y_p(x) = e^x\,(A\cos 2x + B\sin 2x)\Big)(-4)$$

$$\Big[y_p'(x) = e^x\big[(A+2B)\cos 2x + (-2A+B)\sin 2x\big]\Big](-3)$$

$$1\Big[y_p''(x) = e^x\big[(-3A+4B)\cos 2x + (-4A-3B)\sin 2x\big]\Big]$$

$$(\cos x)\cdot\Big(\overbrace{-4A-3A-6B-3A+4B\ \cancel{8}}^{=0}+8\Big)$$

$$+\,(\sin x)\cdot\Big[\underbrace{-4B+6A-3B-4A-3B}_{=0}\Big]$$

$$-10A-2B=-8$$

$$2A-10B=0\ ;\ A=5B$$

$$-50B-2B=-8$$

M $-52B = -8$, ~~$B = \frac{52}{8} = \frac{26}{4} = \frac{13}{2}$~~

$$B = \frac{8}{52} = \frac{4}{26} = \frac{2}{13}$$

$$A = \frac{10}{13}$$

$$y_p(x) = \frac{10}{13} e^x \cos 2x + \frac{2}{13} e^x \sin 2x \; e^x$$

N

$y''(x) - 2y'(x) + y(x) = x;$

$r^2 - 2r + 1 = 0;\quad r = 1$

$y_0(x) = C_1 e^{x} + C_2 \cdot x \cdot e^{x};$

$y_p(x) = Ax + B;$

$y_p'(x) = A;\quad y''(x) = 0$

$\underline{-2A}$ [illegible] $- 2B + \underline{A}x + B = \underline{x}$

~~$A = 1$~~

$-2A + Ax + B = x$

$A = 1;\ B = 2$

$y_G(x) = C_1 e^{x} + C_2 x e^{x} + x + 2$

$$y''(x) - 2y'(x) = x$$

I way $z(x) = y'(x)$

~~$z' - 2z -$~~

$$z'(x) - 2z(x) = x$$

II way

$$y_0''(x) - 2y_0'(x) = 0$$

$$r^2 - 2r = 0; \quad r = 0, \quad r = 2$$

$$y_0(x) = C_1 + C_2 e^{2x}$$

$$y_p(x) = A \cdot x + B$$ ✗

$$y_p'(x) = A; \quad y_p''(x) = 0$$

$$-2A = Ax + B$$

$$Ax = -2A - B$$ no sltn

1P

$$y_p(x) = Ax^2 + Bx$$

$$y'_p(x) = 2Ax + B$$

$$y''_p(x) = 2A$$

$$2A - 2(2Ax + B) = x$$

$$2A - 4Ax - 2B = x$$

$$(2A - 2B) = (4A + 1)x$$

$$\therefore A = -1/4; \quad B = 1/4$$

$$y_G(x) = C_1 + C_2 e^{2x} - \frac{x^2}{4} + \frac{x}{4}$$

$$-2y'_G(x) = (-2)2C_2 e^{2x} + \frac{x}{2} - \frac{1}{2}$$

$$y''_G(x) = \quad 4C_2 e^{2x} + 1/2$$

$$x = x$$

$$0 = 0$$

Q

$y''(x) - 2y'(x) + y(x) = e^x$

$r^2 - 2r + 1 = 0$

$r = 1,$

$y_0(x) = c_1 e^x + c_2 x e^x$

$y_p(x) = A e^x \cdot x^2$

A

October 6

2023

Friday

$$a y''(x) + b y'(x) + c\, y(x) = \overset{B}{f}(x)$$

$f(x) =$ exponential

polynomial

sin/cos

Method of undetermined coefficiets.

IF $f(x) = \left[A \sin(ax) + B \cos(ax)\right] \cdot \sum_{k=0}^{n} C_k X^k \cdot \exp(bx)$

Look for particular solution as

$$y_p(x) = \left(D \sin(ax) + E \cos(ax)\right) \cdot \sum_{k=0}^{n} F_k X^k \cdot \exp(bx) \cdot X^s;$$

$s = 0, 1, 2$

C

RHS	polynomial	y_p = polynomial
	sin/cos	$y_p = A\sin + B\cos$
	e^{ax}	$y_p = Ae^{ax}$

EXAMPLE

D

$$y''(x) = x^2;$$

$$y'(x) = \int x^2 dx = \frac{x^3}{3} + A$$

$$y(x) = \int \frac{x^3}{3} dx = \frac{x^4}{12} + Ax + B$$

Method of undetermined coefficiets

Homogeneous: $y_o''(x) = 0;$

$$y_o = e^{\lambda x}; \lambda^2 = 0;$$

$$\lambda = 0;$$

$$y_o = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

$$= C_1 + C_2 X$$

Inhomogeneous

$$y_p = Ax^{2+1+1} + Bx^{2+1} + Cx^{\cancel{2}+1}$$

E

$$y_p = Ax^4 + Bx^3 + Cx^2$$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y_p'' = 12Ax^2 + 6Bx + 2C$$

$$= x^2$$

$$B = C = 0; \; A = 1/12$$

$$y_G(x) = \underbrace{C_1 + C_2 x}_{\text{general sltn to } y''=0} + \underset{\substack{\uparrow \\ \text{particular sltn}}}{\frac{x^4}{12}}$$

Superposition principle

F

① $ay''(x) + by'(x) + cy(x) = f_1(x) + f_2(x)$

② $y''(x) - 3y'(x) - 4y(x) = e^{-x} + 4x^2 \underbrace{-8e^{x}\cos 2x}$

$$\boxed{-8e^{x}\cos 2x}$$

③ $y_0'' - 3y_0' - 4y_0 = 0$

④ $y_1'' - 3y_1' - 4y_1 \cancel{'} = e^{-x}$ no prime

⑤ $y_2'' - 3y_2' - 4y_2 = 4x^2$

⑥ $y_3'' - 3y_3' - 4y_3 = -8e^{x}\cos 2x$

$\cdots$ y_0

$\underline{\quad}$ y_1

$=$ y_2

y_3 $\equiv$

⑦ $y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x)$

⑧ $= c_1e^{-x} + c_2e^{2x} \underline{- \frac{1}{5}xe^{-x}} \underline{\underline{- x^2 + \frac{3}{2}x - \frac{13}{8}}}$

⑨ $\underline{\underline{\underline{+ \frac{10}{13}e^{x}\cos 2x + \frac{2}{13}e^{x}\sin 2x}}}$

6

$$a y''(x) + b y'(x) + c y(x) = \sum_{i=1}^{n} f_i(x)$$

Homogeneous: $a y_0'' + b y_0' + c y_0 = 0$; $y_0 = \ldots$

1st RHS $a y_1'' + b y_1' + c y_1 = f_1(x)$

2nd RHS $a y_2'' + b y_2' + c y_2 = f_2(x)$

— — — —

nth RHS $a y_n'' + b y_n' + c y_n = f_n(x)$;

2

$$a y''(x) + b y'(x) + c y(x) = \sum_{i=1}^{n} f_i(x)$$

$$y(x) = \sum_{k=0}^{n} y_k(x);$$

Variation of a parameter.

$$y''(x) + p(x)y'(x) + g(x)y(x) = f(x);$$

inhomogeneous eqn

$$y_0''(x) + p(x)y_0'(x) + g(x)y_0(x) = 0;$$

$$y_0(x) = C_1 y_1(x) + C_2 y_2(x)$$

with $W[y_1(x), y_2(x)] \neq 0$

$\forall x;$

I

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p'(x) = \underbrace{u_1'(x)y_1(x) + u_2'(x)y_2(x)}_{=0} + u_1(x)y_1'(x) + u_2(x)y_2'(x);$$

Choose: $\underline{u_1'(x)y_1(x) + u_2'(x)y_2(x)} = 0;$

Simplifies calculations:

(x

$$y_p''(x) = u_1'(x)y_1'(x) + u_2'(x)y_2'(x) + u_1(x)y_1''(x) + u_2(x)y_2$$

$$u_1'y_1' + u_2'y_2' + \underline{u_1y_1''} + \underline{\underline{u_2y_2''}} + p(x)\left(\underline{u_1y_1'} + \underline{\underline{u_2y_2'}}\right)$$

$$+ g(x)\left(\underline{u_1y_1} + \underline{\underline{u_2y_2}}\right) = f(x)$$

$\underline{\quad\quad} = 0$ b/c y_1 is a sltn to $y'' + py' + gy = 0$

$\underline{\underline{\quad\quad}} = 0$ – y_2 – " – $\underline{\quad\quad}$

D

$$\left.\begin{aligned} u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= f(x) \\ u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0 \end{aligned}\right\}$$ Solve for $u_1(x), u_2(x)$

$$u_2'(x) = -\frac{1}{y_2(x)} u_1'(x)\, y_1(x)$$

$$u_1'(x)y_1'(x) - \frac{y_2'(x)}{y_2(x)} u_1'(x)\, y_1(x) = f(x)$$

$$u_1'(x)\left[y_1'(x) - \frac{y_2'(x)}{y_2(x)} y_1(x) \right] = f(x)$$

$u_1'($

$$u_1'(x) = \frac{f(x)}{y_1'(x) - \frac{y_2'(x)}{y_2(x)} y_1(x)} = \frac{f(x)\, y_2(x)}{y_2(x)y_1'(x) - y_2'(x)y_1}$$

$$W(x) = \mathrm{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

$$= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) = W(x)$$

$$u_1'(x) = -\frac{f(x)\, y_2(x)}{W(x)}$$

$$u_1(x) = -\int_0^x \frac{f(s)\, y_2(s)\, ds}{W(s)}$$

$$u_2'(x) = -\frac{1}{y_2(x)}\, u_1'(x)\, y_1(x) = \frac{f(x)\, y_2(x)\, y_1(x)}{W(x)\, y_2(x)}$$

$$= \frac{f(x)\, y_1(x)}{W(x)} \; ; \quad u_2(x) = \int_0^x \frac{f(s)\, y_1(s)\, ds}{W(s)}$$

$$\left.\begin{aligned} u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) &= 0 \\ u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) &= f(x) \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_1(x) &= -\int_0^x \frac{f(s)\,y_2(s)\,ds}{w(s)} \\ u_2(x) &= \int_0^x \frac{f(s)\,y_1(s)\,ds}{w(s)} \end{aligned}}$$

$$y''(x) + y(x) = 2\cos x$$ L

Homogeneous eqn $y_0''(x) + y_0(x) = 0$

$y_1(x) = \sin(x)$; $y_1'(x) = \cos(x)$

$y_2(x) = \cos(x)$; $y_2'(x) = -\sin(x)$

$$W(x) = \mathrm{Det}\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$= -\sin^2(x) - \cos^2(x) = -1$$

$$u_1(x) = -\int dx \, \frac{2\cos^2(x)}{-1} = 2\int dx \cos^2 x$$

$2\cos^2 x = 1 + \boxed{2}\cos x$;

$u_1(x) = \int dx \left(1 + \boxed{2}\cos x\right)$ ~~$= x + 2$ [illegible]~~

$u_2(x) = \int dx$

M

$$u_1(x) = \int dx \left(1 + \cos 2x\right) = x + \frac{\sin 2x}{2} + A$$

$$u_2(x) = \int dx \, \frac{f(x)\, y_1(x)\, dx}{W(x)} =$$

$$= \int \frac{2 \cos x \sin x}{-1} dx = -\int \sin 2x \, dx =$$

$$= \frac{\cos 2x}{2} + B$$

$$y(x) = \left(x + \frac{\sin 2x}{2} + A\right) \sin x$$

$$+ \left(\frac{\cos 2x}{2} + B\right) \cos x$$

Ⓐ

Oct 10

Variation of a parameter

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = f(x) \quad (*)$$

with

corresponding homogeneous eq.:

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0;$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x);$$

$y_p(x)$ to $(*)$ as

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x);$$

$$u_1(x) = -\int_0^x \frac{y_2(s)\,f(s)\,ds}{W(s)}$$

$$u_2(x) = \int_0^x \frac{y_1(s)\,f(s)\,ds}{W(s)}$$

Ⓑ Given

$$y'(x) + p(x)\,y(x) = g(x) \quad (*)$$

If y_0 is a sltn to

$$y'(x) + p(x)\,y(x) = \text{[illegible]}$$

Find sltn to $(*)$ as

$$y(x) = u(x)\,y_0(x)$$

Ⓒ Example

$$y''(x) + y(x) = 2\cos x \qquad (*)$$

$$y_0(x) = C_1 \sin x + C_2 \cos x;$$

Look for sltn of (*)

$$y(x) = u_1(x)\sin x + u_2(x)\cos x$$

$$\boxed{y(x) = A\sin x + \tilde{B}\cos x + x\sin x}$$

$$u_1 = x + \frac{\sin 2x}{2} + A$$

$$u_2 = \left(\frac{1}{2}(1 + \cos 2x) + B\right)$$

$$u_2 = -\frac{1}{2}\cos 2x + \tilde{B}$$

$$y(x) = \left(A + x + \frac{\sin 2x}{2}\right)\cdot \sin x$$

$$+ \left(\tilde{B} + \frac{1}{2}\cos 2x\right)\cos x$$

$$= \cos x$$

Ⓓ undetermined coefficients

$$y''(x) + y(x) = 2\cos x$$

$$y_p(x) = A x \sin x + x B \cos x$$

Ⓔ Example:

$$y''(x) + y(x) = \tan x;$$

$$y_p(x) = u_1(x)\sin x + u_2(x)\cos x;$$

$$W(x) = -1;$$

$$\sin^2 x + \cos^2 x = 1$$

$$-\sin^2(x) =$$

$$= \cos^2 x - 1$$

$$u_1(x) = -\int_0 \frac{y_2(x) f(x) dx}{w(x)} =$$

$$= -\int \frac{\cos x \tan x \, dx}{-1} =$$

$$= \int \sin(x) dx = -\cos x + C_1;$$

$$u_2(x) = \int \frac{y_1(x) f(x) dx}{w(x)} = -\int \sin x \tan x \, d$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x}{\cos x} dx - \int \frac{dx}{\cos x}$$

Ⓕ

$$u_2 = \sin(x) \int \frac{dx}{\cos x} + B$$

$$\int \frac{dx}{\cos x} = \log\left(\tan(x) + \frac{1}{\cos x} \right)$$

$$\frac{d}{dx} \log\left(\tan x + \frac{1}{\cos x} \right) \dot{-}$$

$$= \frac{1}{\tan x + \frac{1}{\cos^2 x}} \cdot$$

× cos(x)

$$u_2 = \sin x - \log\left(\tan x + \frac{1}{\cos x} \right) + B$$

$$y(x) = C_1 \sin x + C_2 \cos x$$

$$\underline{\cos x \cdot \sin x} + \left(\underline{\sin} x - \log\left(\tan x + \frac{1}{\cos x} \right) \right) \cdot \cos(x)$$

$$\boxed{y(x) = C_1 \sin x + C_2 \cos x - \log\left(\tan x + \frac{1}{\cos x} \right)}$$

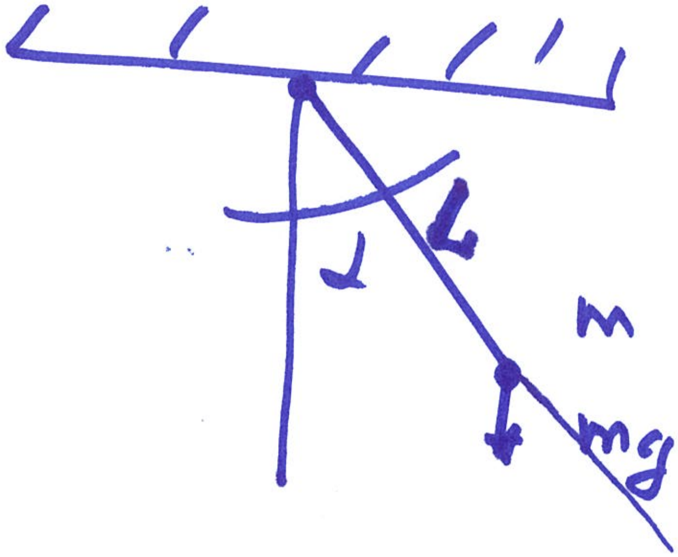
Ⓖ $\frac{d}{dx}\left(\log\left(\tan x + \frac{1}{\cos x}\right)\right)$

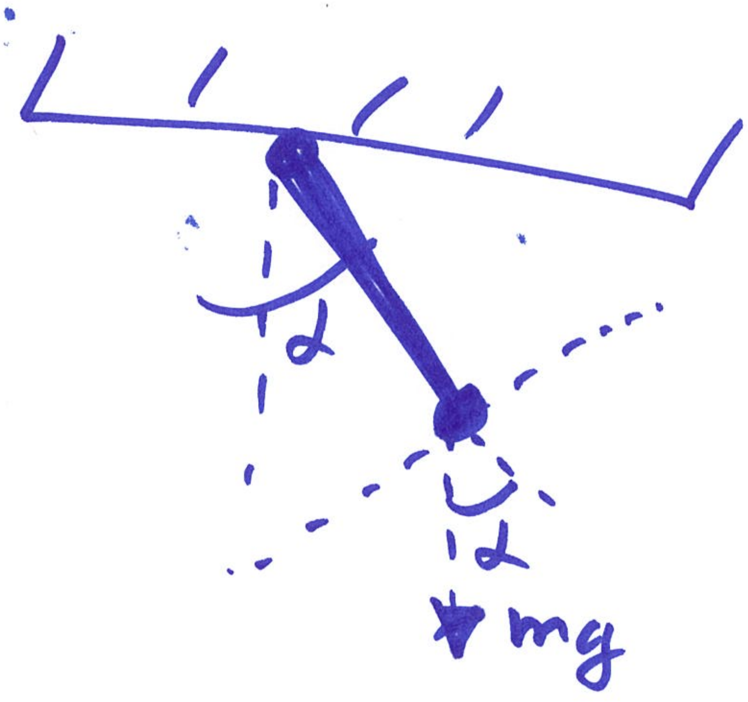
$$= \frac{\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}}{\tan x + \frac{1}{\cos x}} =$$

$$= \frac{1 + \sin x}{\sin x \cos x + \cos x} = \frac{1}{\cos x}$$

Ⓗ

Harmonic oscillations





$$mL\ddot{\alpha} + mg\sin\alpha = 0$$

I

Small oscillations

$|\alpha| << 1$

$\sin\alpha \approx \alpha$.

$$L\ddot{\alpha}(t) + g\alpha(t) = 0$$

Add external force: $F\cos\omega t$

Add friction $+\gamma\dot{\alpha}$

$$L\ddot{\alpha} + \gamma\dot{\alpha} + g\cdot\alpha = F\cos\omega t$$

$$\underbrace{\ddot{\alpha}}_{\downarrow\, x} + \underbrace{\boxed{\frac{\gamma}{L}}}_{\to b\cdot 2}\dot{\alpha} + \underbrace{\boxed{\frac{g}{L}}}_{\to \omega_0^2}\alpha = \underbrace{\boxed{\frac{F}{L}}}_{\to f}\cos\omega t$$

$$\ddot{x} + b\dot{x} + \omega_0^2 x = f\cos\omega t$$

J

$$f(x+\Delta) = f(x) + f'(x)\Delta + f''(x)\Delta^2/2 + \dots$$

$$\sin(0+\Delta) = \sin(0) + \left(\frac{d}{dx}\sin x\right)\Big|_{x=0} \cdot \Delta + \left(\frac{d^2}{dx^2}\sin x\right)\Big|_{x=0} \cdot \Delta^2/2 + \dots$$

$$\approx \Delta$$

K

$$\ddot{x}(t) + 2b\,\dot{x}(t) + \omega_0^2\, x(t) = f\cos\omega t$$

- "Free oscillator": $b = f = 0$

$$\ddot{x}_0(t) + \omega_0^2 x_0(t) = 0$$

$$[\omega_0^2] = \frac{1}{\text{Time}^2}; \quad [g] = \frac{\text{Length}}{\text{Time}^2}$$

$$[g/L] = \frac{1}{\text{Time}^2}$$

$$x_0(t) = e^{rt}; \quad \ddot{x}_0(t) = r^2 e^{rt}$$

$$r^2 + \omega_0^2 = 0; \quad r = \pm i\omega_0$$

$$x_0(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$x_0(t) = R\cos(\omega_0 t - \varphi)$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

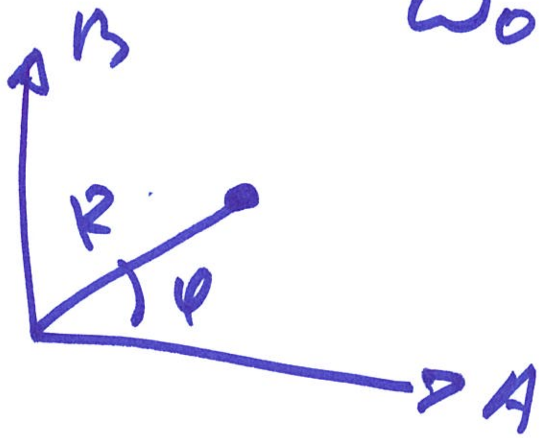
$$x_0(t) = R\cos(\omega_0 t - \varphi)$$

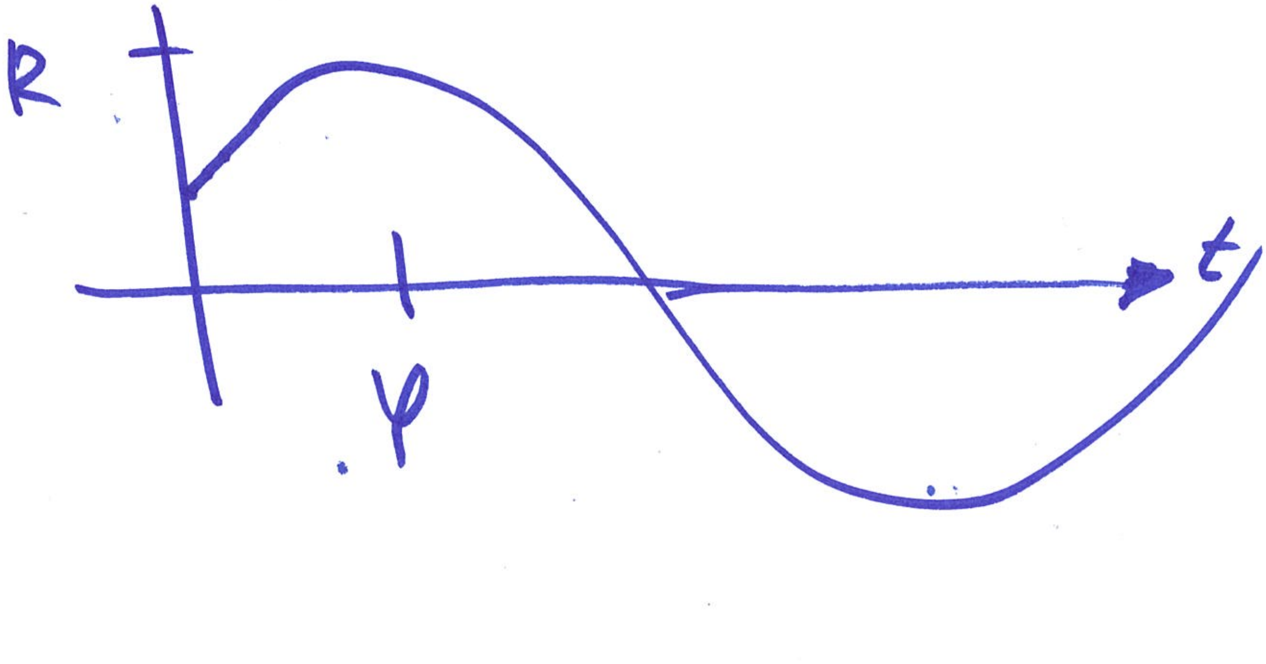
$$= \underbrace{R\cos\varphi}_{A}\cdot\cos\omega_0 t + \underbrace{R\sin\varphi}_{B}\sin\omega_0 t$$

Period: $T = \frac{2\pi}{\omega_0}$

R - amplitude

φ - phase





M

$$x(t) = R \sin(\omega_0 t - \psi)$$

N

- Damped oscillator

$$b > 0 ; f = 0 ;$$

$$\ddot{x}(t) + 2b\dot{x}(t) + \omega_0^2 x(t) = 0$$

$$x(t) = e^{rt}$$

$$r^2 + 2br + \omega_0^2 = 0$$

$$r_{1,2} = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2}$$

$$= -b \pm \sqrt{b^2 - \omega_0^2} < 0$$

ⓐ $b^2 > \omega_0^2$: overdamped oscillator

Two real roots

$$x(t) = Ae^{\left(-b - \sqrt{b^2 - \omega_0^2}\right)t} + Be^{\left(-b + \sqrt{b^2 - \omega_0^2}\right)t}$$

ⓒ under damped oscillator.

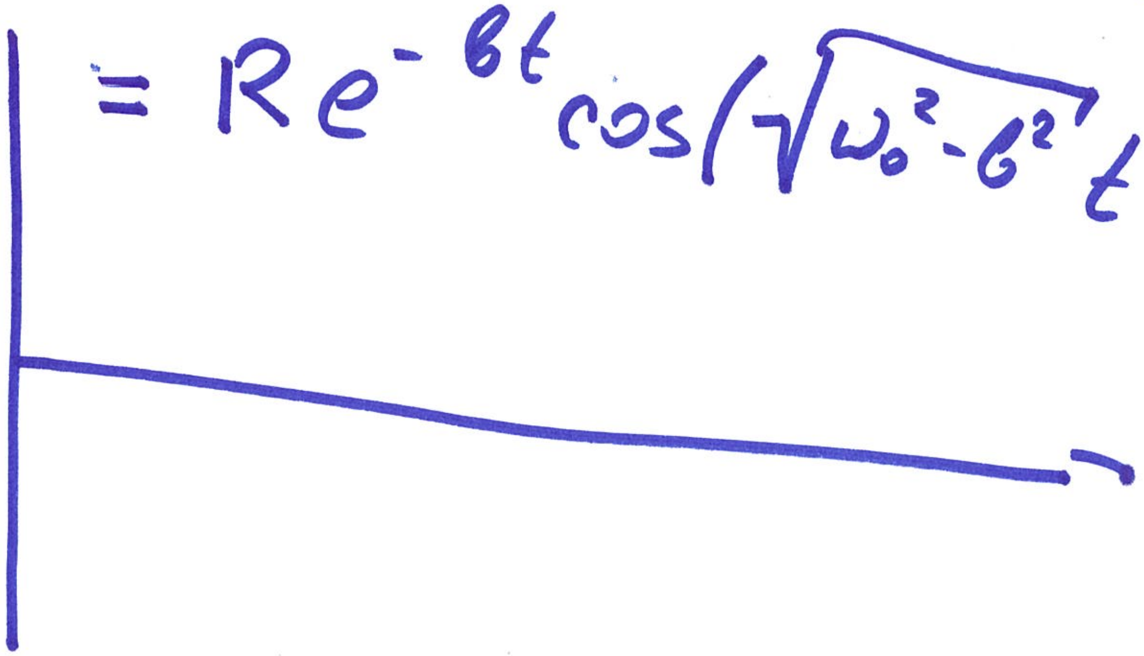
$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \qquad \beta^2 < \omega_0^2$$

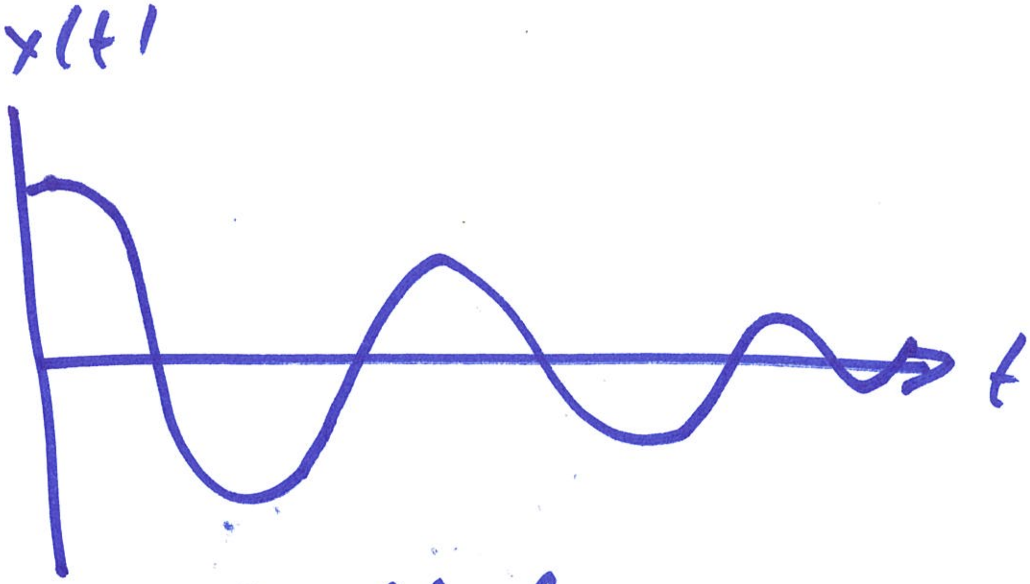
$$= -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

$$x(t) = A \cdot e^{-\beta t} \cdot \cos(\sqrt{\omega_0^2 - \beta^2} \cdot t)$$

$$+ B \cdot e^{-\beta t} \cdot \sin(\sqrt{\omega_0^2 - \beta^2}\, t)$$

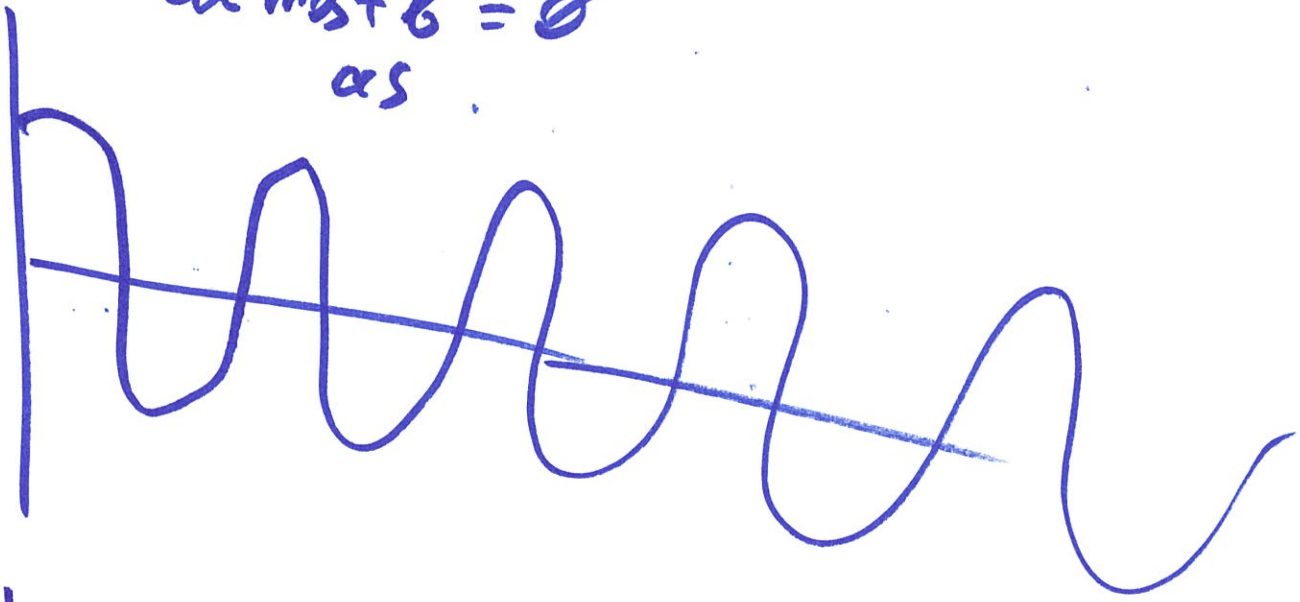
$$= R e^{-\beta t} \cos(\sqrt{\omega_0^2 - \beta^2}\, t - \varphi)$$

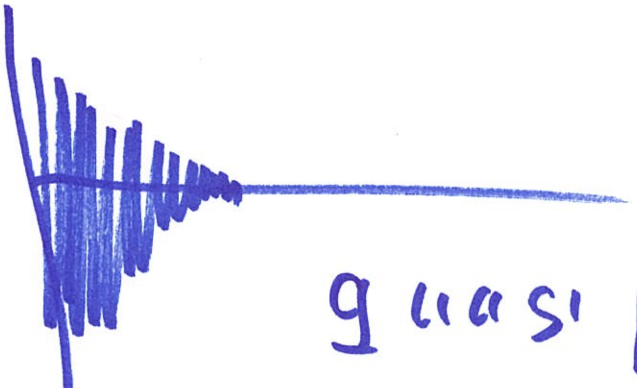




small b

almost b = 0 as





quasi period:

$$T \approx \frac{2\pi}{\sqrt{\omega_0^2 - b^2}}$$

A

October 13, 2023

Harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = F\cos\omega t$$

$x \equiv x(t);$

friction

elastic force

external force

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = f\cos\omega t.$$

(I) Free oscillator $\beta = f = 0$

(II) Damped harmonic oscillator

$f = 0; \; \beta > 0$

underdamped

critically

overdamped

Forced oscillator: $\beta = 0$; C

$$\ddot{x}(t) + \omega_0^2 x(t) = f\cos(\omega t) \quad (1)$$

Nonresonant $\omega \neq \omega_0$

- Homogeneous equation $x_0(t) = R\cos(\omega_0 t - \varphi)$

- Particular solution

$$x(t) = A\cos\omega t + B\sin\omega t \quad (2)$$

$$\ddot{x}(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t; \quad (3)$$

(2), (3) → (1):

$$\underline{-A\cdot\omega^2\cos\omega t} - B\omega^2\sin\omega t$$
$$+\underline{\omega_0^2 A\cos\omega t} + \omega_0^2 B\sin\omega t$$
$$= \underline{f\cos\omega t}$$

$\Downarrow [\ ]_1$

D

$$(\cos\omega t)[A(\omega_0^2-\omega^2)-f]$$

$$+(\sin\omega t)[B(\omega_0^2-\omega^2)]=0;$$

$[\ ]_2$

$$[\ ]_1=[\ ]_2=0$$

$$B(\omega_0^2-\omega^2)=0$$

$$A(\omega_0^2-\omega^2)=f$$

$$B=0;\ A=\frac{f}{\omega_0^2-\omega^2}$$

$$x(t)=\underbrace{R\cos(\omega_0 t-\varphi)}_{\text{free oscillations}}+\frac{f}{\omega_0^2-\omega^2}\cos(\omega t)$$

forced oscillations

E

Forced oscillator

resonant case

$$\ddot{x}(t) + \omega_0^2 x(t) = f\cos(\omega_0 t)$$

$$x_p(t) = At\cos\omega_0 t + Bt\sin\omega_0 t;$$

$$\dot{x}_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$$
$$- At\omega_0\sin\omega_0 t + Bt\omega_0\cos\omega_0 t$$

$$\ddot{x}_p(t) = -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t$$
$$- At\omega_0^2\cos\omega_0 t - Bt\omega_0^2\sin\omega_0 t$$

$$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t$$

$$-At\omega_0^2 \cos\omega_0 t - Bt\omega_0^2 \sin\omega_0 t$$

$$+\omega_0^2 At\cos\omega_0 t + Bt\omega_0^2 \sin\omega_0 t$$

$$= f\cos\omega_0 t$$

$$(\sin\omega_0 t)\left[-2A\omega_0\right]$$

$$+\left[\cos\omega_0 t\right](2B\omega_0 - f) = 0$$

$$A = 0; \quad B = \frac{f}{2\omega_0}$$

~~$x(t) = R\cos\omega_0 t$~~

$$x(t) = R\cos(\omega_0 t - \varphi) + \frac{f\cdot t}{2\omega_0}\sin(\omega_0 t)$$

G

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = f\cos(\omega t)$$

$$x_p(t) = A\cos\omega t + B\sin\omega t;$$

$$\dot{x}_p(t) = -A\omega \sin\omega t + B\omega\cos\omega t;$$

$$\dot{x}_p(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\ddot{x}_p(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t;$$

$$-\underline{A\omega^2\cos\omega t} - B\omega^2\sin\omega t - 2\beta A\omega\sin\omega t + \underline{2\beta B\overset{\omega}{\vee}\cos\omega t}$$

$$+\underline{\omega_0^2 A\cos\omega t} + \omega_0^2 B\sin\omega t = \underline{f\cos\omega t}$$

$$[\cos\omega t]\left[-A\omega^2 + 2\beta B\omega + \omega_0^2 A - f\right]$$

$$+[\sin\omega t]\left[-B\omega^2 - 2\beta A\omega + \omega_0^2 B\right] = 0$$

$$A(\omega_0^2 - \omega^2) + 2\beta B\omega = f$$

$$-2\beta\omega A + (\omega_0^2 - \omega^2)B = 0$$

$$B = \frac{2\beta\omega A}{\omega_0^2 - \omega^2}$$

H

$$A(\omega_0^2-\omega^2)+(2\delta\omega)^2 A \frac{1}{\omega_0^2-\omega^2} = f$$

$$A\left[(\omega_0^2-\omega^2)^2+(2\delta\omega)^2\right] = f(\omega_0^2-\omega^2);$$

$$A = \frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(2\delta\omega)^2}$$

$$B = \frac{2\delta\omega f}{(\omega_0^2-\omega^2)^2+(2\delta\omega)^2}$$

$$x_p(t) = R\cos(\omega t-\varphi)$$

$$R = \sqrt{A^2+B^2} = \sqrt{\frac{f^2(\omega_0^2-\omega^2)^2+(2\delta\omega f)^2}{\left((\omega_0^2-\omega^2)^2+(2\delta\omega)^2\right)^2}}$$

I

$$R = f \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

$$\varphi = \mathrm{ArcTan}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$x(t) = \tilde{R} e^{-\beta t} \cos\left(\sqrt{\omega_0^2 - \beta^2}\, t - \tilde{\varphi}\right)$$

free oscillations

$$+ R\cos(\omega t - \varphi)$$

forced oscillati
oscillations

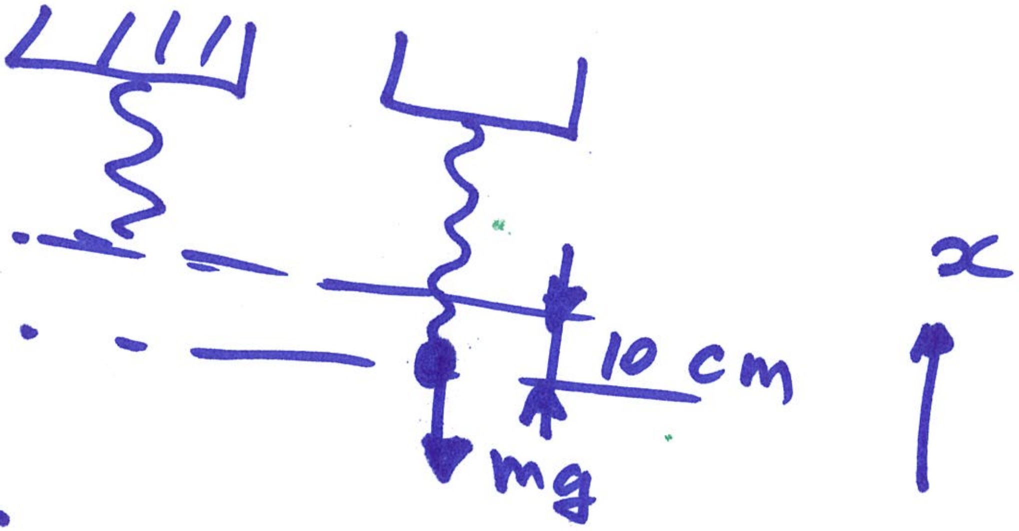
$\tilde{R}, \tilde{\varphi}$ are two arbitrary constants to satisfy

IC

K

Mass = 5 kg

stretches spring by 10 cm



$$\bar{F} = -mg + (\Delta x)k = 0$$

$$k = \frac{mg}{\Delta X} = \frac{5\,kg \cdot 10\ meters/sec^2}{10\ cm}$$

$$= 5\,kg \cdot 100/sec^2$$

2 Newtons for 4 cm/sec

$$2\ Newton = \gamma \cdot 4\ cm/sec$$

$$= \gamma \cdot 0.04\ meters/sec$$

$$\gamma = \frac{2}{0.04}\ \frac{kg \cdot meter\ sec}{sec^2\ meters} = 50\ \frac{kg}{sec}$$

L

$$k = 500 \frac{\text{kg}}{\text{sec}^2}$$

$$\gamma = 50 \frac{\text{kg}}{\text{sec}}$$

$$F(t) = 10 \sin\left(\frac{t}{2\,\text{sec}}\right) \text{ Newton}$$

$$m\ddot{x} + \gamma\dot{x} + xk = F \sin \omega t$$

$$5\,\text{kg} \cdot \ddot{x}(t) + 50 \frac{\text{kg}}{\text{sec}} \dot{x}(t) + 500 \frac{\text{kg}\, x(t)}{\text{sec}^2}$$

$$= 10 \sin(t/2\,\text{sec}) \text{ Newton}$$

$$x(t=0) = 0$$

$$\dot{x}(t=0) = 0.03 \frac{\text{meters}}{\text{sec}}$$

=

L

$$k = 500 \frac{kg}{sec^2}$$

$$\gamma = 50 \frac{kg}{sec}$$

$$F(t) = 10 \sin\left(\frac{t}{2sec}\right) \text{ Newton}$$

$$m\ddot{x} + \gamma\dot{x} + xk = F \sin \omega t$$

$$5kg \cdot \ddot{x}(t) + 50 \frac{kg}{sec}\dot{x}(t) + 500 \frac{kg\, x(t)}{sec^2}$$

$$= 10 \sin(t/2sec) \text{ Newton}$$

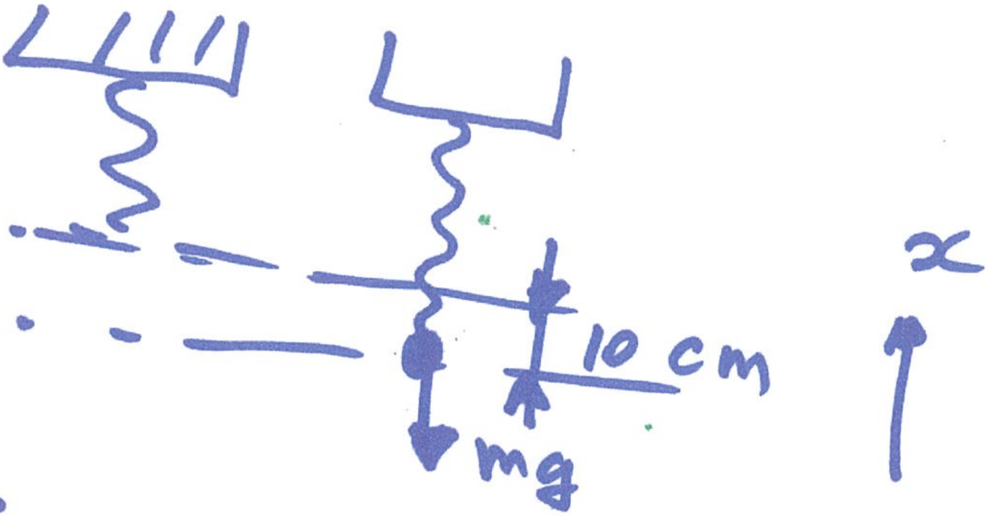
$$x(t=0)=0$$

$$\dot{x}(t=0) = 0.03 \frac{meters}{sec}$$

K

Mass = 5 kg

stretches spring by 10 cm



$$\bar{F} = -mg + (\Delta x)k = 0$$

$$k = \frac{mg}{\Delta X} = \frac{5\,kg \cdot 10 \text{ meters/sec}^2}{10 \text{ cm}}$$

$$= 5\,kg \cdot 100/sec^2$$

2 Newtons for 4 cm/sec

$$2 \text{ Newton} = \gamma \cdot 4 \text{ cm/sec}$$

$$= \gamma \cdot 0.04 \text{ meters/sec}$$

$$\gamma = \frac{2}{0.04} \frac{kg \cdot meter \; sec}{sec^2 \; meters} = 50 \frac{kg}{sec}$$

A

OCTOBER 17, 2023

$$5\,\text{kg}\,\ddot{x}(t) + 50\,\frac{\text{kg}}{\text{sec}}\,\dot{x}(t) + 500\,\frac{\text{kg}}{\text{sec}^2}\,x(t)$$

dimension

$$= 10 \sin\left(\frac{t}{2\,\text{sec}}\right) \text{Newton}$$

$[x] = L \quad [t] = T$

time

$x = \tilde{x} L \;; \quad t = \tilde{\tau}\cdot T \quad [\tau] = [\tilde{x}] = 1$

L, T are characteristic length, time

$[L] =$ meters; $[T] =$ seconds.

$$\frac{d}{dt} = \frac{dt}{d\tau}\frac{d}{d\tau} = \frac{1}{T}\frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{d\tau^2}$$

$$5\,\text{kg}\cdot L\,\frac{1}{T^2}\,\frac{d^2}{d\tau^2}\tilde{x}(\tau) + 50\,\frac{\text{kg}}{\text{sec}}\cdot\frac{L}{T}\,\frac{d}{d\tau}\tilde{x}(\tau)$$

$$+ 500\,\frac{\text{kg}}{\text{sec}^2}\cdot L\cdot\tilde{x}(\tau) = 10 \sin\left(\frac{\tau T}{2\,\text{sec}}\right)\frac{\text{kg}\cdot\text{meter}}{\text{sec}^2}$$

$L = 1\,\text{meter}; T = 1\text{sec}$

$$5\,\text{kg}\cdot\frac{\text{meter}}{\text{sec}^2}\frac{d^2}{dT^2}\tilde{x}(T)$$

$$+50\,\frac{\text{kg}}{\text{sec}}\,\frac{\text{meter}}{\text{sec}}\,\frac{d}{dT}\tilde{x}(T)$$

$$+500\,\frac{\text{kg}}{\text{sec}^2}\cdot\text{meter}\cdot\tilde{x}(T) =$$

$$10\sin\left(\frac{T\cdot 1\text{sec}}{2\text{sec}}\right)\cdot\frac{\text{kg}\cdot\text{meter}}{\text{sec}^2}$$

$$5\ddot{\tilde{x}} + 50\dot{\tilde{x}} + 500\tilde{x} = 10\sin(T/2)$$

$$\dot{\tilde{x}} = \frac{d}{dT}\tilde{x}; \quad \ddot{\tilde{x}} = \frac{d^2}{dT^2}\tilde{x}(T)$$

Example

C

$$\ddot{u} + 2\dot{u} + 2u = 5\sin t$$

$$u(t=0) = \dot{u}(t=0) = 0$$

Find $u(t)$ ∀t, does it depend upon initial conditions?

Soln

- Homogeneous: $\ddot{u}_0 + 2\dot{u}_0 + 2u_0 = 0$

$$r^2 + 2r + 2 = 0$$

$$r_{12} = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= -1 \pm i$$

$$u_0 = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t;$$

- $u_p = A\sin t + B\cos t$ $\qquad \cos t \equiv c$

$\dot{u}_p = A\cos t - B\sin t$ $\qquad \sin t \equiv s$

$\ddot{u}_p = -A\sin t - B\cos t$

$$\underline{-As} - \utilde{Bc} + \utilde{2A}c \underline{- 2Bs} + \underline{2As} + 2\utilde{Bc}$$

$$= \underline{5s}$$

$(\sin t)[+A - 2B - 5] +$

$+ (\cos t)[B + 2A] = 0$

$\left.\begin{aligned} -A + 2B &= -5 \\ 2A + B &= 0 \end{aligned}\right]$ $B = -2A$

$-A - 4A = -5 = -5A$

~~$A = 5/3$~~ $A = 1$

D

$B = -2$

$u_1(t) = u_{1o}(t) + u_{1p}(t)$

$= C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$

$+ \sin t - 2 \cos t;$

$u_1(t=0) = C_1 - 2 = 0 \Rightarrow C_1 = 2$

$\dot{u}_1(t) = -C_1 e^{-t} \cos t - C_1 e^{-t} \sin t - C_2 e^{-t} \sin t$

$+ C_2 e^{-t} \cos t + \cos t + 2 \sin t;$

$\dot{u}_1(t=0) =$ [illegible] $-2 + C_2 + 1 = 0$

$C_2 = 1;$

E

$$u(t) = 2e^{-t}\cos t + e^{-t}\sin t$$
$$+ \sin t - 2\cos t;$$

$$\lim_{t\to\infty} u(t) = \sin t - 2\cos t,$$

independent of IC

Ⓕ

Example

Find general solution

$\ddot{x} + 4x = 3\cos t$; $x_0 = e^{rt}$

$r^2 + 4 = 0$

$r = \pm 2i$

Homogeneous $\ddot{x}_0 + 4x_0 = 0$

$x_0(t) = A\cos 2t + B\sin 2t$

$x_p = C\cos t + D\sin t$

$\ddot{x}_p = -C\cos t - D\sin t$

$-C\cos t - D\sin t + 4C\cos t + 4D\sin t = 3\cos t$;

$(\cos t)[3C - 3] + (\sin t)[3D] = 0$

$C = 1$; $D = 0$;

$x(t) = A\cos 2t + B\sin 2t + \cos t$

$$\ddot{x} + \underline{\underline{\omega_0}}^2 x = 0$$

G

Euler equation

$$a x^2 y''(x) + b x y'(x) + c y(x) = 0$$

Linear second order ODE with nonconstant coefficients.

Look for solution as

$$y(x) = x^r$$

$$y'(x) = r x^{r-1}$$

$$y''(x) = r \cdot (r-1) x^{r-2}$$

$$a x^2 \cdot r \cdot (r-1) x^{r-2} + b \cdot x \cdot r \cdot x^{r-1} + c x^r = 0;$$

$\underline{x > 0}$ $x^r \neq 0$

$$\boxed{a \cdot r(r-1) + b \cdot r + c = 0}$$

I

$a r^2 + (b-a) r + c = 0$

- $r_1 \neq r_2$; $\operatorname{Im} r_1 = \operatorname{Im} r_2 = 0$

  $(b-a)^2 > 4ac$

  $y(x) = C_1 \cdot x^{r_1} + C_2 \cdot x^{r_2}$

- $(b-a)^2 = 4ac$, $r_1 = r_2$

  $y(x) = \ldots$

- $(b-a)^2 < 4ac \Rightarrow$ two complex roots

K

$y(x) = |x|^r$

L

Two equal real roots

$$0 = x^2 y''(x) + (2\beta+1) x\, y'(x) + \beta^2 y(x)$$

$$y(x)=x^r;\quad r\cdot(r-1)+(2\beta+1)r+\beta^2=0$$

$$r^2+2\beta+\beta^2=0 \Rightarrow r=-\beta$$

[redacted]

$$\left(x^2\frac{d^2}{dx^2}+(2\beta+1)x\frac{d}{dx}+\beta^2\right)y(x)=0$$

$$\shortparallel$$

$$\left(x\frac{d}{dx}+\beta\cdot x\right)\left(x\frac{d}{dx}+\beta x\right)y(x)=0$$

To be continued

$$x\frac{d}{dx}\left(\left(x\frac{d}{dx}+\beta x\right)y(x)\right)=$$

$$=\left(x\left(\frac{d}{dx}+x^2\frac{d^2}{dx^2}+x\beta\right)y(x)+\beta x\frac{d}{dx}\right)y(x)$$

$$=\left(x^2\frac{d^2}{dx^2}+(2\beta+1)\frac{d}{dx}+\beta\right)y(x)$$

$$\left(x\frac{d}{dx}+\beta\right)\left(x\frac{d}{dx}+\beta\right)y(x)$$

$$=\left(x\frac{d}{dx}+x^2\frac{d^2}{dx^2}+2x\beta\frac{d}{dx}+\beta^2\right)y(x)$$

$$=\left(x^2\frac{d^2}{dx^2}+(2\beta+1)x\frac{d}{dx}+\beta^2\right)y(x)$$

$$=x^2y''(x)+(2\beta+1)x\,y'(x)+y(x)\beta^2=0$$

$\left(x\frac{d}{dx}+\beta\right)y(x)=z(x)$; $\quad xz'+\beta z=0$

$$\frac{dz}{z}=-\frac{\beta\,dx}{x}\Rightarrow \ln z=-\beta\ln x=\ln\frac{1}{x}$$

$z(x)=1/x^\beta$; $\quad x\,y'(x)+\beta y(x)=z(x)=1/x^\beta$

~~$\frac{d}{dx}(x$~~ $\quad y'+\frac{\beta}{x}y=\frac{1}{x^{\beta+1}}$; $\mu(x)=x^\beta$

$$x^\beta y'+\beta x^{\beta-1}y=1/x$$

$$\frac{d}{dx}\left(x^\beta y(x)\right)=1/x\Rightarrow y(x)=x^{-\beta}\ln x;$$

M

$$EQ = \left(x\frac{d}{dx} + \beta \cancel{x}\right)\left(x\frac{d}{dx} + \beta \cancel{x}\right)y(x)$$

$$= x^2\frac{d^2}{dx^2} + (2\beta+1)\frac{d}{dx} + \cancel{\beta x y}$$

$$+ \cancel{\beta x^2 \frac{d}{dx}} + \beta^2 \cancel{x^2}\, y(x)$$

IF $r_1 = r_2 = r$

$$y(x) = C_1 x^r + C_2 x^r \ln x;$$

Example

$$x^2 y''(x) + 5x\, y'(x) + 4y(x) = 0;$$

$$r(r-1) + 5r + 4 = 0$$

$$r^2 + 4r + 4 = 0; \quad r_1 = -2 = r_2$$

$$y(x) = C_1|x|^{-2} + C_2|x|^{-2}\ln|x|;$$

Two complex roots:

$r_{1,2} = a \pm i b$

$y(x) = C_1 \cdot x^{a+ib} + C_2 \cdot x^{a-ib}$

$x^{a+ib} = x^a \cdot x^{ib} =$

$= x^a \exp(\ln(x^{ib}))$

$= x^a \exp(i b \ln x) =$

$= x^a (\cos(\ln x) + i \sin(b \ln x))$

$b\cdot$

$= x^a \cos(b \ln x) + i x^a \sin(b \ln x)$

$y(x) = C_1 x^{a+ib} + C_2 x^{a-ib}$

$= C_1 x^a (\cos(b \ln x) + i \sin(b(\ln$

$+ C_2 x^a (\cos(b \ln x) - i \sin(b \ln x$

$$y(x) = \boxed{(C_1 + C_2)}\, x^a \cos(b \ln x)$$

$$+ \underbrace{\boxed{i\,(C_1 - C_2)}}_{D_2}\, x^a \sin(b \ln x)$$

October 20

A

Lecture

Euler equation

$$a x^2 y''(x) + x b y'(x) + c y(x) = 0;$$

$$y(x) = x^r;$$

$$a r(r-1) + b r + c = 0$$

- 2 real distinct roots

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

- repeated double root

$$y(x) = C_1 x^r + C_2 x^r \ln x;$$

- 2 complex roots

$$r_{1,2} = a \pm i b$$

$$y(x) = C_1 x^a \cos(b \ln x) + C_2 x^a \sin(b \ln x);$$

Double real root

$$x^2 y''(x) + (2b+1)x\,y'(x) + b^2 y(x) = 0$$

$$y(x) = x^r$$

$$r(r-\underline{1}) + (2b\underline{+1})r + b^2 = 0$$

$$r^2 + 2br + b^2 = 0$$

$$(r+b)^2 = \underline{0}, \quad r = -b;$$

$$\left(x^2 \frac{d^2}{dx^2} + (2b+1)\,x\frac{d}{dx} + b^2\right) y(x) = 0$$

$$\left(x\frac{d}{dx} + b\right)\left(x\frac{d}{dx} + b\right) y(x) = \underline{0}.$$

$$\left[x\frac{d}{dx}\left(x\frac{d}{dx}\right) + bx\frac{d}{dx} + bx\frac{d}{dx} + b^2\right] y(x) = 0$$

$$x\frac{d}{dx}\left(x\frac{d}{dx}\right) = x\frac{d}{dx} + x^2\frac{d^2}{dx^2}$$

$$\left(x\frac{d}{dx}+6\right)\underbrace{\boxed{\left(x\frac{d}{dx}+6\right)y(x)}}_{z(x)}=0$$

c

$$x\frac{d}{dx}z(x)+6z(x)=0$$

$$\ln z=\int\frac{dz}{z}=-\int\frac{6\,dx}{x}=-6\ln x+C$$

$$=\ln\frac{1}{x^6}+C$$

$$=\ln\frac{\tilde{C}}{x^6}$$

$$z(x)=\frac{\tilde{C}}{x^6}$$

$$x\frac{dy}{dx}+6y(x)=\frac{\tilde{C}}{x^6}$$

$$\frac{dy}{dx}+\frac{6}{x}y(x)=\frac{\tilde{C}}{x^{6+1}}\ ;\ \mu=x^6$$

$$\underbrace{x^6 \frac{dy}{dx} + 6x^{6-1} y(x)}_{\frac{d}{dx}\left(x^6 y(x)\right)} = \frac{\tilde{C}}{x} \qquad D$$

$$\frac{d}{dx}\left(x^6 y(x)\right) = \frac{\tilde{C}}{x}$$

$$x^6 y(x) = \int \frac{\tilde{C}}{x} dx = \tilde{C} \ln x + \tilde{D}$$

$$y(x) = \tilde{C} x^{-6} \ln x + \tilde{D} x^{-6}$$

$x^2 y''(x) + xy'(x) + y(x) = 0;$ $\Leftarrow$

$r(r-1) + r + 1 = 0$

$r^2 = -1; \quad r = \pm i;$

$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x);$

$\Leftarrow$ Example

F

Example

$$x^2 y''(x) + 3x\, y'(x) + 5y(x) = 0$$

$$r(r-1) + 3r + 5 = 0$$

$$r^2 + 2r + 5 = 0$$

$$r_{12} = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$$y(x) = \frac{c_1}{x}\cos(2\ln x) + \frac{c_2}{x}\sin(2\ln$$

$$ax^2 + bx + c = 0$$

G

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{\left(\sqrt{b^2 - 4ac} - b\right)\left(\sqrt{b^2 - 4ac} + b\right)}{2a\left(\sqrt{b^2 - 4ac} + b\right)}$$

$$= \frac{b^2 - 4ac - b^2}{2a\left(\sqrt{b^2 - 4ac} + b\right)}$$

$$= \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

H1

Second order linear ODE.

$$y(x) = \overset{u_1(x)}{C_1}\, y_1(x) + \overset{u_2(x)}{c_2}\, y_2(x);$$

$$W[y_1(x), y_2(x)] \neq 0;$$

constant coefficients

$$ay'' + by' + cy = 0$$

$$y = e^{rx}; \; ar^2 + br + c = 0$$

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$= C_1 e^{(\mathrm{Re}\, r)x} \cos((\mathrm{Im}\, r)x)$$

$$+ c_2 e^{(\mathrm{Re}\, r)x} \sin((\mathrm{Im}\, r)x)$$

$$= R \cos(\omega t - \varphi)$$

$$= R \sin(\omega t - \psi)$$

I

$$a y''(x) + b y'(x) + c\, y(x) = f(x)$$

if $f(x) =$ polynomial, sin, cos,
exp

Method of undetermined
coefficients;

Variation of a
parameter

$$u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$$

$$u_1'(x) y_1'(x) + u_2'(x)\, y_2'(x) = f(x).$$

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)}\, dx$$

$$u_2(x) = \int \frac{y_1(x) f(x)\, dx}{W(x)}$$

Harmonic oscillator

Wolfram Mathematica

Euler Equation

Systems of linear equations.

K

$$(r-2)(r+1)=0$$

$$r^2-r-2=0;$$

$$y''(x)-y'(x)-2y(x)=0;$$

$$y(x)=c_1e^{2x}+c_2e^{-x};$$

$$z(x)=y'(x); \quad \boxed{y'(x)=z(x)}$$

$$y''(x)=y'(x)+2y(x)=z(x)+2y(x)$$

$$\boxed{z'(x)=2y(x)+z(x)}$$

$$\boxed{\begin{aligned} y'(x)&=z(x)\\ z'(x)&=2y(x)+z(x)\end{aligned}}$$

L

The n'th order ordinary differential equation is equavalent to a system of n 1st order ode's.

M

$$\left.\begin{aligned} y'(x) &= z(x) \\ z'(x) &= 2y(x) + z(x) \end{aligned}\right\}$$

$$\frac{d}{dx}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$$

$$y(x) = c_1 e^{2x} + c_2 e^{-x}$$

$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x}$$

$$\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} = c_1 e^{2x}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \,//$$

Vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.

$\Big(y($

N

- Vectors $\underline{x}_1$ and $\underline{x}_2$ are linearly independent if

$$c_1 \underline{x}_1 + x_2 c_2 = 0 \iff c_1 = c_2 = 0$$

$$\underline{x}_2 = - \frac{c_1}{c_2} x_1$$

- Vectors $x_1, x_2 .. x_n$ are linearly independent if

$$\sum_{k=1}^{n} c_k x_k = 0 \iff c_1 = c_2 = .. = c_n = 0$$

$$\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}=\begin{pmatrix}2\\ 4\end{pmatrix}=2\begin{pmatrix}1\\ 2\end{pmatrix}$$

0

$$\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}=-\begin{pmatrix}1\\ -1\end{pmatrix}$$

$\begin{pmatrix}1\\ 2\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}$ with eigenvalue = 2

$\begin{pmatrix}1\\ -1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}$ with eigenvalue = -1

$$y(x) = C_1 \cdot e^{\lambda_1 x}\,\underline{v_1} + C_2 e^{\lambda_2 x}\,\underline{v_2}$$

where v_1, v_2 are eigenvalues of matrix with λ_1, λ_2 being corresponding, eigenvalues.

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

φ

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdot\ \cdot & a_{1n} \\ a_{21} & a_{12} & \cdot\ \cdot & a_{2n} \\ \vdots & & & \\ a_{n1} & - & - \ - & a_{nn} \end{pmatrix}$$

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t);$$

R

$\underline{\underline{A}} \in R^{n \times n}$

vector $\underline{v}$ is called eigenvector and λ is called eigenvalue if

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v};$$

How to find eigenvalues?

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{v} = 0$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = 0$$

where $\underline{\underline{I}}$ is identity matrix;

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$ 5

$$\left.\begin{array}{l} b = -a \\ 2a + b = -b \end{array}\right\} \quad \begin{array}{l} b = -a \\ 2a = -2b \end{array}$$

$a = 1;\ b = -1;$

Eigenvalue -1 has eigenvector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 2;$

T

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

Find eigenvectors and eigenvalues:

$$\det(A - \lambda I) =$$

$$= \det\left[\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] =$$

$$= \det\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) - 2$$

$$= \lambda^2 - \lambda - 2 = 0$$

$$\lambda_1 = -1; \quad \lambda_2 = 2;$$

$$\underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{matrix} 0 & 1 \\ 2 & \end{matrix}$$

October A

24

2023

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{x} \in R^n, \quad \underline{\underline{A}} \in R^{n \times n}$$

$$\underline{x} \equiv \underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}; \quad \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Eigenvector, eigenvalue

$\underline{v}$ λ

$$\underline{\underline{A}}\,\underline{v} = \lambda\,\underline{v}$$

IF $\underline{\underline{A}}$ is $R^{n \times n}$ then there are n eigenvalues.

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

C

$$\Updownarrow$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{v} = 0$$

has solutions iff

$$\mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \leftarrow \text{equation for } \lambda$$

n values of λ

for each one you solve

$$\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i$$

Example D

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

find eigenvalues and eigenvectors.

$$0 = \mathrm{Det}\,(\underline{\underline{A}} - \lambda \underline{\underline{I}}) =$$

$$= \mathrm{Det}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \mathrm{Det}\begin{pmatrix} 1-\lambda, & 0 \\ 0, & 2-\lambda \end{pmatrix} =$$

$$= (1-\lambda)(2-\lambda) = 0 \Rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \end{array}$$

E

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$0 = \det(A - \lambda I) =$$

$$= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= (\lambda - a)(\lambda - d) - bc =$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$\underline{\lambda_1 = 1}$ $\quad \lambda_2 = 2 \quad$ $A = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ $\quad$ F

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\underline{v}_1 = 1\cdot\underline{v}_1$$

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix} = \begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix}$$

$$v_{11} = v_{11} \; ; \; v_{11} = 1$$

$$2v_{12} = v_{12} \quad v_{12} = 0$$

$$\underline{v}_1 = \begin{pmatrix}1\\ 0\end{pmatrix} ; \; v_1 = \begin{pmatrix}2\\ 0\end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$v_{21} = 2 v_{21} \qquad v_{21} = 0$$
$$2 v_{22} = 2 v_{22} \qquad v_{22} = 1$$

$$\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

has eigenvalue 1 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and -//- 2 -//- $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

H

$\underline{v}_1$ and $\underline{v}_2$ are

linearly independent

$v_1 = \begin{pmatrix}1\\0\end{pmatrix}$; $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$;

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 = 0$$

$$c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix} = 0 \iff c_1 = c_2 = 0$$

$$A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

I

Eigenvalues:

$$(2-\lambda)(2-\lambda)-\overset{1}{\cancel{2}}=0;$$

$$4+\lambda^2-4\lambda-1=0$$

$$\lambda^2-4\lambda+3=0$$

$$\lambda_2=\frac{4\pm\sqrt{16-12}}{2}=1,3;$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a + b = a \Rightarrow a = -b$$

$$a + 2b = b \qquad a = -b$$

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \lambda_1 = 1$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 3\begin{pmatrix} c \\ d \end{pmatrix}$$

$$2c + d = 3c, \quad c = d$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \lambda_1 = 1;\ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda_2 = 3;\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}}$$

L

$c_1 \underline{v}_1 + c_2 \underline{v}_2 =$

$$= \begin{pmatrix} c_1 + c_2 \\ c_2 - c_1 \end{pmatrix} = 0 \Leftrightarrow c_1 = c_2 = 0$$

M

$$A = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix};$$

$\lambda_1 = 25;\ \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 25;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 25;$

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\begin{pmatrix} 25-\lambda & 0 \\ 0 & 25-\lambda \end{pmatrix}$$

$$= (25-\lambda)^2 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = 25$$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; \quad \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 25\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$25 v_1 = 25 v_1$$

$$25 v_2 = 25 v_2$$

$$\Rightarrow \begin{pmatrix} v_1 = v_1 \\ v_2 = v_2 \end{pmatrix} \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Any vector in R^2 will be an eigenvector of this matrix;

LET eigenvalue is repeated:

$$\lambda_1 = \lambda_2$$

$$\underline{\underline{A}}\, \underline{v}_1 = \lambda_1 \cdot v_1$$

$$\underline{\underline{A}} \cdot v_2 = \lambda_1 \cdot v_2$$

$$x = c_1 v_1 + c_2 v_2$$

$$\underline{\underline{A}}\, \underline{x} = \underline{\underline{A}}\left(c_1 v_1 + c_2 v_2\right) =$$

$$= c_1 A v_1 + c_2 A v_2 =$$

$$= c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = \lambda_1 \left(c_1 v_1 + c_2 v_2\right)$$

1º

$$\underline{\underline{A}} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

$$0 = Det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = Det \begin{pmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 - 3 \cdot 0 = (\lambda - 2)^2$$

$\lambda_1 = \lambda_2 = 2$

Deffective matrix

$$\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} 2a + 3b = 2a \\ 2b = 2b \end{matrix} \quad \Rightarrow \quad \begin{matrix} b = 0 \\ b = b \end{matrix}$$

$$\begin{pmatrix} a = 1 \\ b = 0 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \underline{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}$$ Q

$$0 = Det(A - \lambda I) =$$

$$= Det\left[\begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= Det\begin{bmatrix} 1-\lambda & 2 \\ -1/2 & 1-\lambda \end{bmatrix} =$$

$$= (1-\lambda)^2 + 2 \cdot 1/2 =$$

$$= (1-\lambda)^2 + 1 = 0$$

$$(\lambda - 1)^2 = -1 \Rightarrow \lambda - 1 = \pm i$$

$$\lambda = 1 \pm i$$

IF $A \in R^{2\times 2}$ R

and it has complex eigenvalues, then

$$\lambda_1 = a + i b$$

$$\lambda_2 = a - i b;$$

$$\lambda_1^* = \lambda_2$$

$$A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}$$

R

$\lambda_1 = (1+i)$

$\lambda_2 = (1-i)$

$$\boxed{\underline{v} = \begin{pmatrix} 2 \\ i \end{pmatrix}}$$

$\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (1+i)\begin{pmatrix} a \\ b \end{pmatrix}$$

$a + 2b = (1+i)a = a + ia \,;\, 2b = ia$

~~aa~~

$-1/2\, a + b = (1+i)b = b + ib \,;\, -\frac{a}{2} = ib$

$\boxed{(2b = ia) \Rightarrow}$ $\quad a = 2$

$b = i$

$$\boxed{2b = i\boxed{a}}$$

$$-\frac{a}{2} = ib \Rightarrow \boxed{-a = 2i}\,b$$

$$a = -2i$$

$$b = 1 \qquad \Rightarrow v = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$v = \begin{pmatrix} ? \\ i \end{pmatrix}$$

$$v = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

T

$$A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}$$

has eigenvalue $\lambda_1 = 1+i$.

with eigenvector

$$v_1 = \begin{pmatrix} 2 \\ i \end{pmatrix}$$

and complex conjugate eigenvalue $\lambda_2 = 1 - i$

and corresponding eigenvector

$$v_2 = v_1^* = \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

Let $A \in R^{n \times n}$,

Let λ be a complex eigenvalue $\lambda_1 = \gamma + i\omega$ with eigenvector $\underline{v}_1 = \underline{a} + i\,\underline{b}$

Then $\lambda_2 = \lambda_1^* = \gamma - i\omega$ is also an eigenvalue with eigenvector $\underline{v}_2 = \underline{a} - i\,\underline{b}$

$$\left(\underline{\underline{A}} \cdot \underline{v} = \lambda \cdot \underline{v}\right)^*$$

$$\underline{\underline{A}}^* \underline{v}^* = \lambda^* v^* \Rightarrow$$

$$\underline{\underline{A}}\, \underline{v}^* = \lambda^* \underline{v}^*$$

$\lambda_1 = 1 + i$

$\lambda_2 = \lambda_1^* = 1 - i$

$\lambda_2 = \bar{\lambda}_1$ ← another notation for CC

CC ≡ complex conjugate

$n \times n$ matrix A has w

n eigenvalues.

→ n distinct real eigenvalues then there n distinct eigenvectors

→ if real eigenvalues are repeated m times, then

either have m distinct eigenvectors

or

have less them m eigenvectors => deffective

X

complex eigenvalues
with complex eigenvecto

Then

Complex Conjugate
eigenvalues are
still eigenvalues with
corresponding conjugate
eigenvectors

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

Let λ be eigenvalue
and v be an eigenvector

$$\underline{\underline{A}}\,\underline{v} = \lambda\,\underline{v}$$

Then $x(t) = \underline{v} \cdot e^{\lambda t};$

$$\dot{x}(t) = \boxed{\underline{v} \cdot \lambda \cdot e^{\lambda t}}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\left(\underline{v}\, e^{\lambda t}\right)$$

$$= e^{\lambda t}\,\underline{\underline{A}}\,\underline{v} = \boxed{e^{\lambda t}\,\lambda\,\underline{v}}$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

n eigenvalues

$$\underline{x}(t) = \sum c_i v_i e^{\lambda_i t}$$

A

October 27

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C

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$$

$$\mathrm{Det}\,(A - \lambda I) =$$

$$\mathrm{Det} \begin{pmatrix} 2-\lambda & 0 \\ 1 & 5-\lambda \end{pmatrix} =$$

$$= (2-\lambda)(5-\lambda) = 0;$$

$$\lambda_1 = 2; \quad \lambda_2 = 5;$$

$\underline{\lambda_1 = 2}$

$$\begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$2a = 2a \qquad a = a$$

$$a + 5b = 2b \qquad a = -3b \qquad \cancel{v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

$$\underline{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}2 & 0\\ 1 & 5\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}-6\\ 2\end{pmatrix}=$$

$$=2\begin{pmatrix}-3\\ 1\end{pmatrix}$$

$$\begin{pmatrix}2 & 0\\ 1 & 5\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=5\begin{pmatrix}a\\ b\end{pmatrix}$$

$$2a=5a \Rightarrow a=0$$

$$a+5b=5b \Rightarrow b=1$$

$$\underline{v}_2=\begin{pmatrix}0\\ 1\end{pmatrix}$$

Nonhomogeneous linear second order equations

E

$$\ell(x)\, y''(x) + p(x) y'(x) + q(x)\, y(x) = f(x) \quad ($$

- Constant coefficients

  $\ell(x) = a; \quad p(x) = b, \; q(x) = c$

- Euler Equation

  $\ell(x) = x^2; \quad p(x) = x; \; q(x) = 1$

F

Undetermined coefficients

IF

$ay''(x) + by'(x) + c\,y(x) =$

=
- polynomial
- exponent
- sin, cos
- or sum of the above

$y_p(x) =$ is the same general form, as RHS.

G

$$f(x) = \sum_{k=0}^{m} a_k x^k e^{\alpha x} \cdot (\sin \beta x + \cos \beta x)$$

$$y_p = \sum_{k=0}^{m} x^k e^{\alpha x} \cdot x^s \cdot (A_k \sin \beta x + B_k \cos \beta x)$$

$s = 0, 1, 2$

so that no terms of a particular solution solve homogeneous equations

Variation of a parameter.

H

y[illegible] Equation (*):

$y_1(x)$ and $y_2(x)$ are linearly independent solutions to (*) with $f(x) \overset{\parallel}{\not\!0}$

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x);$$

$$\left.\begin{aligned} u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) &= 0 \\ u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) &= 0 \end{aligned}\right\}$$

$$u_1(x) = -\int \frac{y_2(x)\,f(x)\,dx}{W(x)}$$

$$u_2(x) = \int \frac{y_1(x)\,f(x)\,dx}{W(x)}$$

$$y'(x) + p(x)\,y(x) = f(x)$$

Homogeneous equation

$$y_0'(x) + p(x)\,y_0(x) = 0$$

$$\int \frac{dy_0}{y_0} = -\int p(x)\,dx$$

$$y_0(x) = e^{-\int p(x)\,dx}$$

$$y_p(x) = u(x)\,e^{-\int p(x)\,dx}$$

$$y_p'(x) = u'(x)\,e^{-\int p(x)\,dx}$$
$$- p(x)\,u(x)\,e^{-\int p(x)\,dx}$$

$$u'(x)e^{-\int p(x)dx}$$

$$- p(x)u(x)e^{-\int p(x)dx}$$

$$+ p(x)u(x)e^{-\int p(x)dx}$$

$$= f(x)$$

$$u'(x) = f(x)e^{\int p(x)dx}$$

$$u(x) = \left(\int f(x)e^{\int p(x)dx}\right)$$

K

$$m\ddot{x} + b\dot{x} + kx = F\cos\omega t$$

$$\ddot{y} + 2\gamma\dot{y} + \omega_0 y = f\cos\omega t$$

- Free oscillator $\gamma = f = 0$
- Damped oscillator : $f = 0$, $\gamma > 0$
  - under damped
    $0 < \gamma < \beta$
  - critically damped
    $\gamma = \beta$
  - overdamped $\gamma > \beta$
- forced - resonant / non resonant
- forced and damped

$y(t) =$ free oscillations

$+$ forced oscillations

$$y_p(x) = R\cdot\cos(\omega t - \varphi)$$

~~$R = \frac{1}{\sqrt{\omega^2 - \omega_0^2}}$~~ $R = \frac{}{\sqrt{\cdots}}$

L

Euler equation

$$a x^2 y''(x) + b x\, y'(x) + c\, y(x) = 0;$$

$$y(x) = x^r$$

$$a r(r-1) + b r + c = 0;$$

→ 2 real distinct roots

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

→ ~~2 re~~ double real root

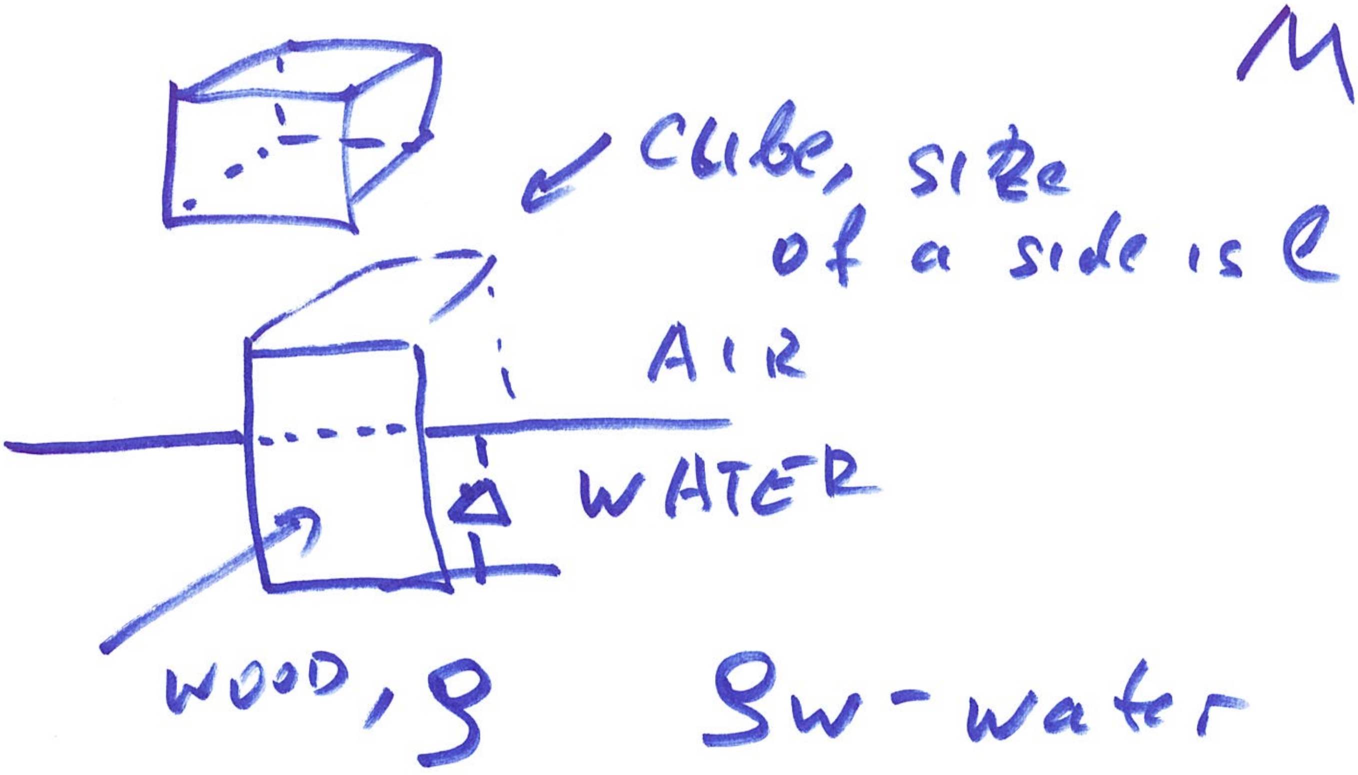
$$r_1 = r_2 = r$$

$$y(x) = C_1 x^r + C_2 x^r \ln x;$$

→ 2 complex roots: $r_{1,2} = a \pm i b$

$$y(x) = C_1 x^a \cos(b \ln x) + C_2 x^a \sin(b \ln x)$$

M

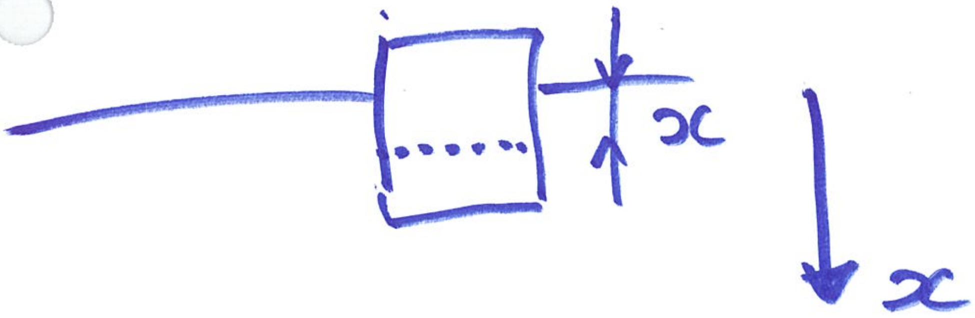


what is $\Delta = ?$

$$g \, \Delta \ell^2 \rho_w = \ell^3 \rho \, g$$

$$\Delta = \frac{\ell^3 \rho}{\ell^2 \rho_w} = \ell \frac{\rho}{\rho_w}$$

N



$$\underbrace{(\rho l^3)}_{\text{mass}}\ddot{x} = -x l^2 \rho_w g$$

$$\rho l^3 \ddot{x} + x l^2 \rho_w g = 0$$

$$\ddot{x} + x\frac{\rho_w g}{\rho l} = 0$$

$$\omega_0 = \sqrt{\frac{\rho_w g}{\rho l}}$$

Solving linear systems of equations.

$\underline{\underline{A}} \in R^{n \times n}$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

① Find eigenvalues:

$$\mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

n eigenvalues.

② Find eigenvectors

solve $\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i$

n eigenvectors. that are $\boxed{\text{dent}}$

linearly indepen

P

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \leftarrow$ linear

$x(t) = \underline{v}_i e^{\lambda_i \cdot t}$

$\underline{\dot{x}} - \underline{\underline{A}}\,\underline{x} = \lambda_i v_i e^{\lambda_i t} -$

$- \underline{\underline{A}}\,\underline{v}_i e^{\lambda_i t} =$

$= \lambda_i v_i e^{\lambda_i t}$

$- \lambda_i v_i e^{\lambda_i t} = 0$

$\swarrow$ eigen pair

IF λ_1, v_1 and $\lambda_2 v_2$ are

2 · eigenpairs

then $y(x) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$

$\dot{y}(x) = c_1 \underline{v}_1 \lambda_1 e^{\lambda_1 t} + c_2 \underline{v}_2 \lambda_2 e^{\lambda_2 t}$

φ

IF

$\lambda_1, \lambda_2 \ldots \lambda_n$ are eigenvalues of $\underline{\underline{A}}$

and $v_1, v_2 \ldots v_n$ are corresponding eigenvectors

General solution of

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$ is

$$x(t) = \sum_{k=1}^{n} c_k \underline{v_k} e^{\lambda_k t}$$

$v_1, v_2 \ldots v_n$ are linearly independent eigenvectors.

$\underline{\dot{x}}(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \underline{x}(t);$

Eigenvalues:

$$\mathrm{Det}\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

$(2-\lambda)^2 - 1 = 0$

$(2-\lambda)^2 = 1; \quad 2-\lambda = \pm 1$

$\lambda = 2 \mp 1 = 1, 3$

<u>eigenvectors</u>

$\boxed{\lambda_1 = 1}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} 2a + b = a \\ a + 2b = b \end{matrix} \Rightarrow a = -b, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 3\begin{pmatrix} c \\ d \end{pmatrix}$$

T

$$\begin{aligned} 2c + d &= 3c \\ c + 2d &= 3d \end{aligned} \Rightarrow c = d$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 1; \; v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 3; \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$= \begin{pmatrix} c_1 e^{t} + c_2 e^{3t} \\ -c_1 e^{t} + c_2 e^{3t} \end{pmatrix}$$

$$\dot{\underline{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \underline{x}$$

$$x(t=0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$x(t=0) = \begin{pmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$-1 = C_1 = -C_2$$

$$x(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

Nov 7 A

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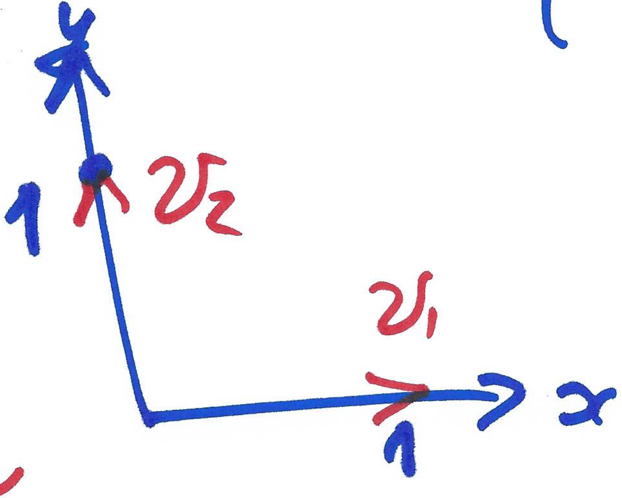
B

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

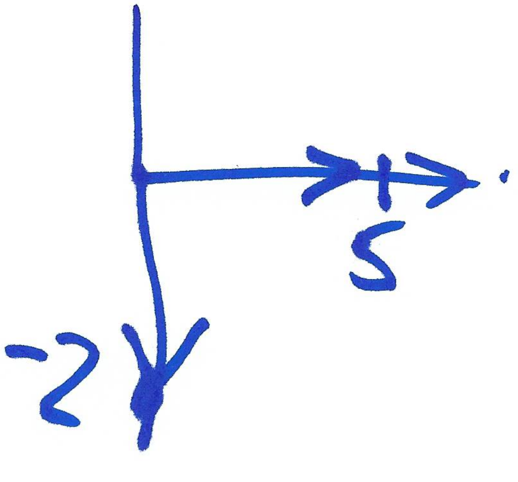
$\lambda_1 = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2 \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\tilde{v}_1 = 5 \cdot v_1$

$\tilde{v}_2 = -2 \cdot v_2$



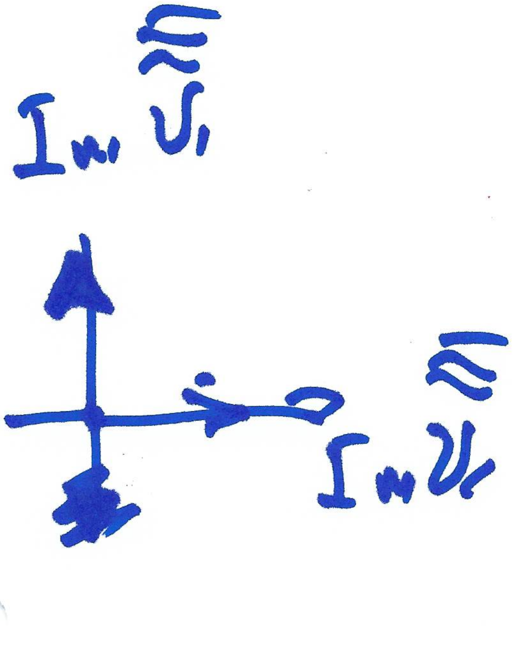
$\tilde{\underline{v}}_1 = i\,\underline{v}_1$ <

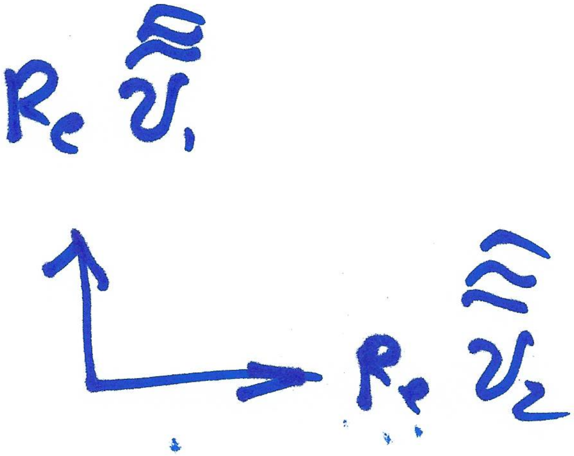
$\tilde{\tilde{\underline{v}}}_2 = i\,\underline{v}_2$

$\tilde{v}_1, \tilde{\tilde{v}}_2$ are eigenvectors

- $\overset{\approx}{\underline{v}}_1 = (22+25i)\,v_1$

  $\overset{\approx}{v}_2 = (27-20i)\,v_2$





D

Deffective matrix

is an nxn matrix with less than n eigenvectors

$A \in R^{2\times 2}$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$;

A has eigenvalue λ and eigenvector $\underline{v}$;

$x(t) = c_1 \underline{v} e^{\lambda t}$;

$x(t) = \underline{v} \cdot t e^{\lambda t} + b e^{\lambda t}$;

$\dot{x}(t) = \underline{v} e^{\lambda t} + \lambda \underline{v} t e^{\lambda t} + b\lambda e^{\lambda}$

$\dot{x}(t) = e^{\lambda t} (\underline{v} + \lambda \underline{v} t + b\lambda)$

$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(\underline{v} t + b) e^{\lambda t}$ ~~$= \underline{\underline{A}}\,\underline{v} \lambda t + A$~~

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(\underline{v}t + \underline{b})e^{\lambda t} =$$

$$= \lambda v t e^{\lambda t} + \underline{\underline{A}}\,\underline{b}\,e^{\lambda t}$$

$$= \dot{x} = e^{\lambda t}(\underline{v} + \lambda \underline{v} t + \underline{b}\lambda)$$

$$\underline{v} + \underline{b}\lambda = \underline{\underline{A}} \cdot \underline{b}$$

or $\boxed{(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{b} = \underline{v}}$

L

F

IF $A \in \mathbb{R}^{3\times 3}$, deffective with one eigenvector

$$\underline{x} = \left(c_1 w + c_2 u t + c_3 v t^2\right) e^{\lambda t}$$

2 eq for u, w

guess

F

Summary

To solve $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$

① Find eigenvalues of

$\underline{\underline{A}}$: $\mathrm{Det}(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

② Find eigenvectors

~~$v_1, v_2, \ldots v_n$~~ v_1, v_2

③
- $\lambda_1 \neq \lambda_2$, $\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$

  Two distinct real eigenvalues

  $\underline{v}_1, \underline{v}_2$ are two linearly independent vectors

  $x(t) = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t}$

- $\lambda_1 = \lambda_2^*$ - two complex conjugate eigenvalues

$$\underline{x}(t) = C_1(\underline{p}\cos\omega t - \underline{q}\sin\omega t)e^{\gamma t}$$
$$+ C_2(\underline{p}\sin\omega t + \underline{q}\cos\omega t)e^{\gamma t}$$

where $\underline{p} = \mathrm{Re}\,\underline{v}$ $\quad \omega = \mathrm{Im}\,\lambda$

$\underline{q} = \mathrm{Im}\,\underline{v}$ $\quad \gamma = \mathrm{Re}\,\lambda$

- one eigenvalue
  - two linearly independent eigenvectors $\underline{v}_1, \underline{v}_2$

    $$x(t) = C_1\underline{v}_1 e^{\lambda t} + C_2\underline{v}_2 e^{\lambda t}$$

  - one eigenvector $\underline{v}$

    Solve $(A - \lambda I)\underline{b} = \underline{v}$

    $$x(t) = C_1\underline{v}e^{\lambda t} + C_2(\underline{v}t + \underline{b})e^{\lambda t}$$

H

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathrm{Det}\,(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda) =$$

$$\lambda_1 = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2 \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 3 \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{x} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

I

Phase Plane.

Plot trajectories $y(x)$;

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \underline{x}$$

$$\underline{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

$$\underline{x}(t=0) = \underline{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\underline{x}(t=0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Solution to IVP

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

with $x(t=0) = x_0$

$y(t=0) = y_0$

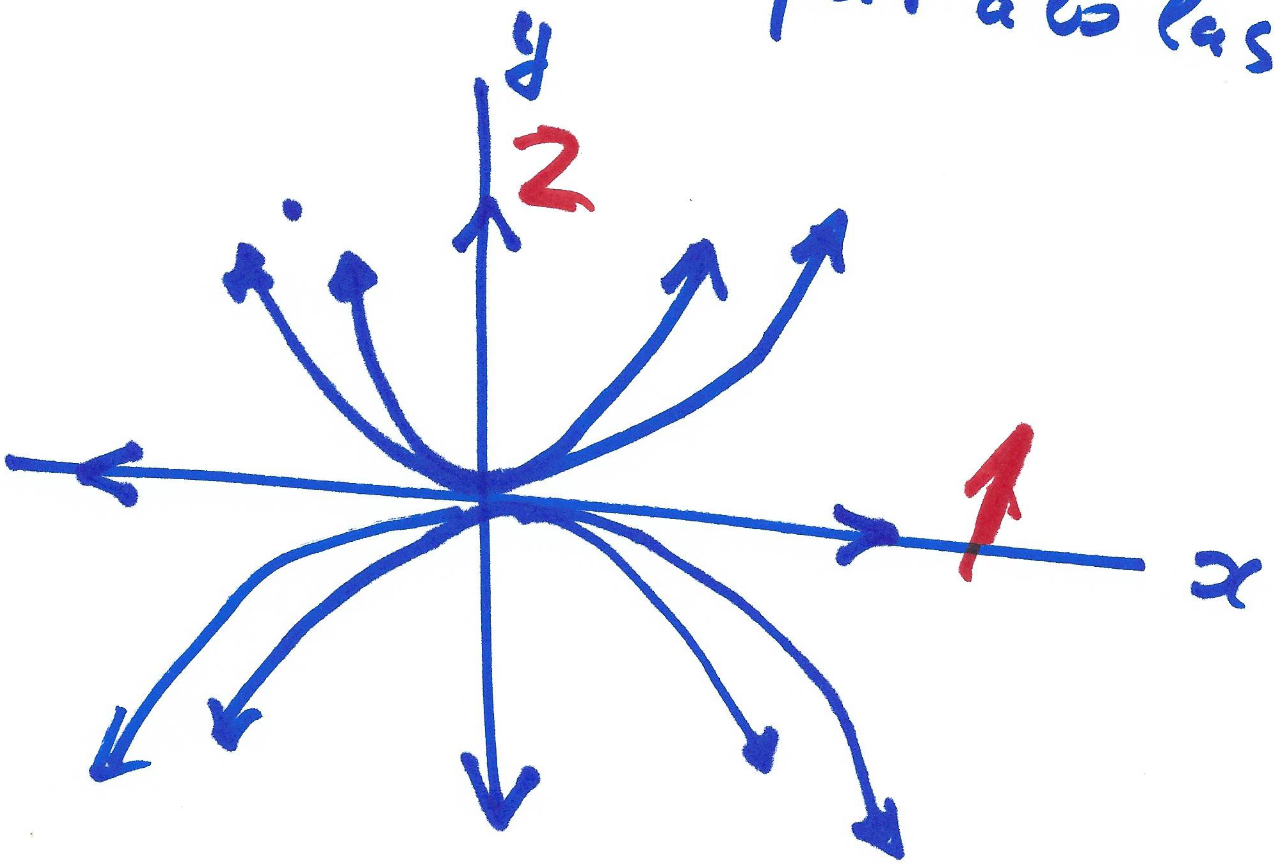
is

$$\underline{x}(t) = \begin{pmatrix} x_0 e^{t} \\ y_0 e^{2t} \end{pmatrix}$$

$x(t) = x_0 e^{t} \Rightarrow e^{t} = \frac{x(t)}{x_0}$ K

$y(t) = y_0 e^{2t} = y_0 \left(\frac{x(t)}{x_0}\right)^2$

Trajectories are parabolas



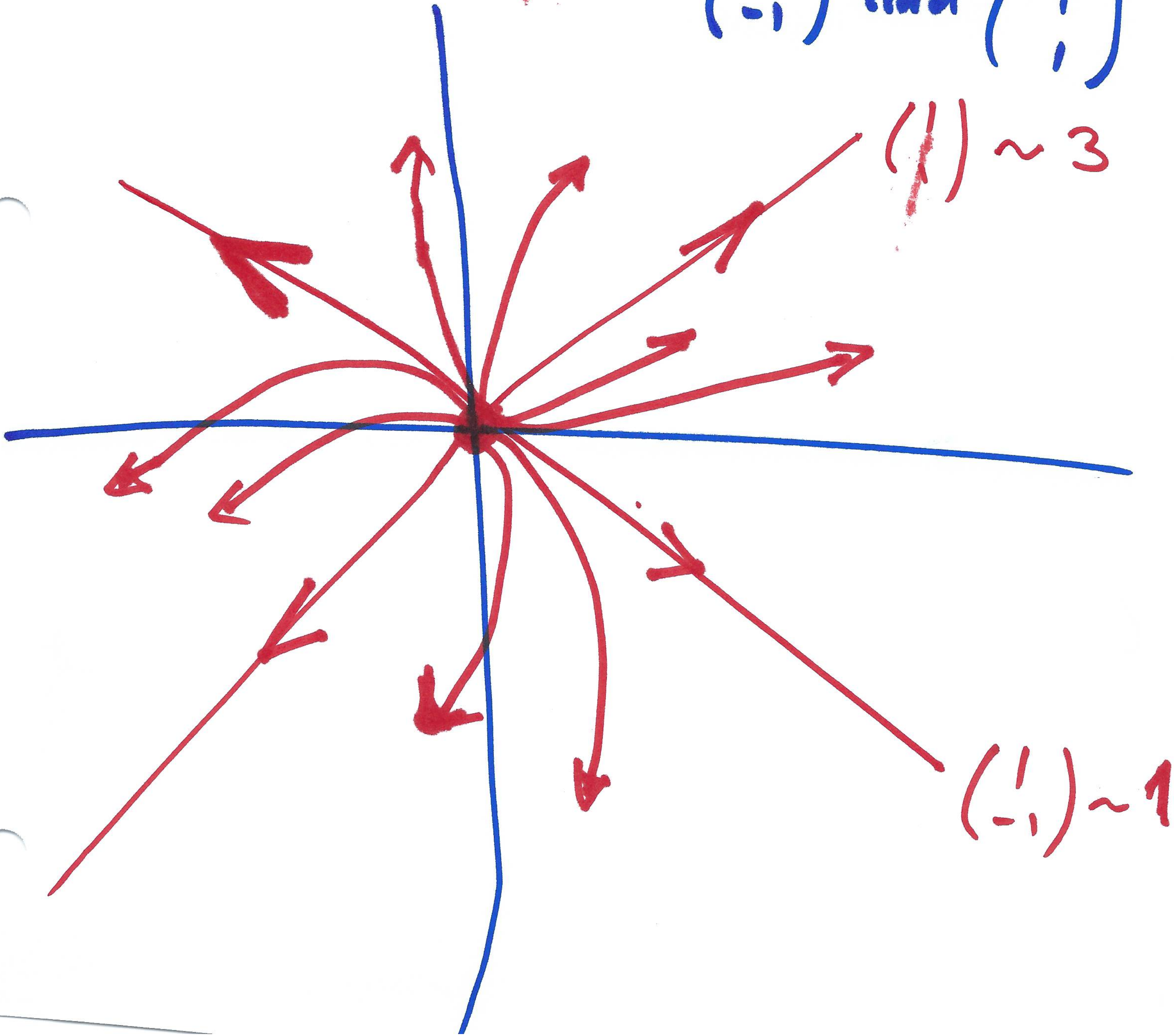
$$\dot{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x(t)$$

L

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

eigenvalues are 1, 3

eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



M

$$\underline{\dot{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \underline{x}$$

$$\underline{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

~~$\underline{x}(t)$~~ $x(t=0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Rightarrow$

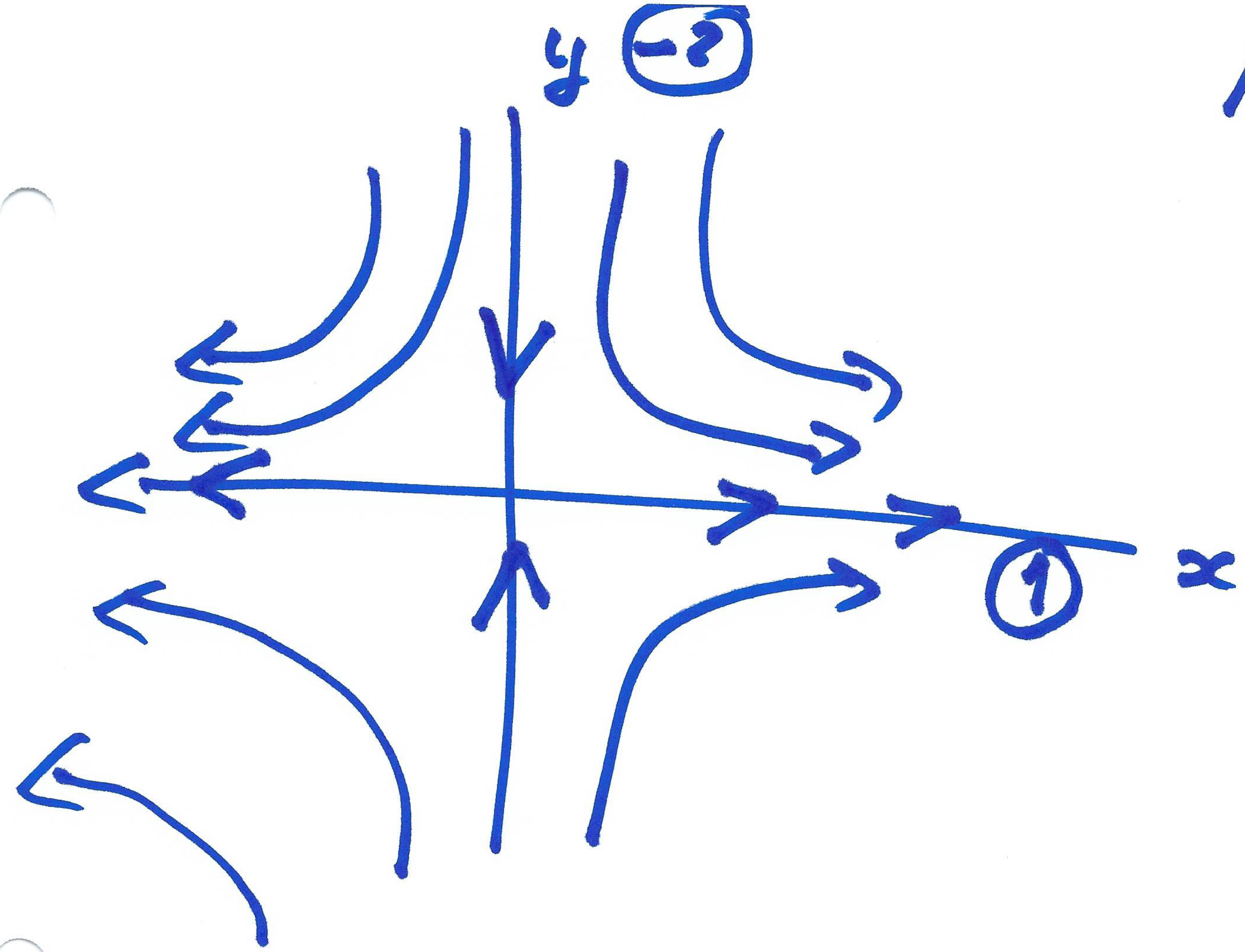
$$\underline{x}(t) = \begin{pmatrix} x_0 e^{t} \\ y_0 e^{-2t} \end{pmatrix}$$

$$e^{t} = x(t) / x_0$$

$$e^{-2t} = \left(\frac{x_0}{x(t)}\right)^2$$

$$y(x) = y_0 \left(\frac{x_0}{x(t)}\right)^2 = y_0 \left(\frac{x_0^2}{(x(t))^2}\right)$$

λ



∠

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$\lambda_1 = 2; \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

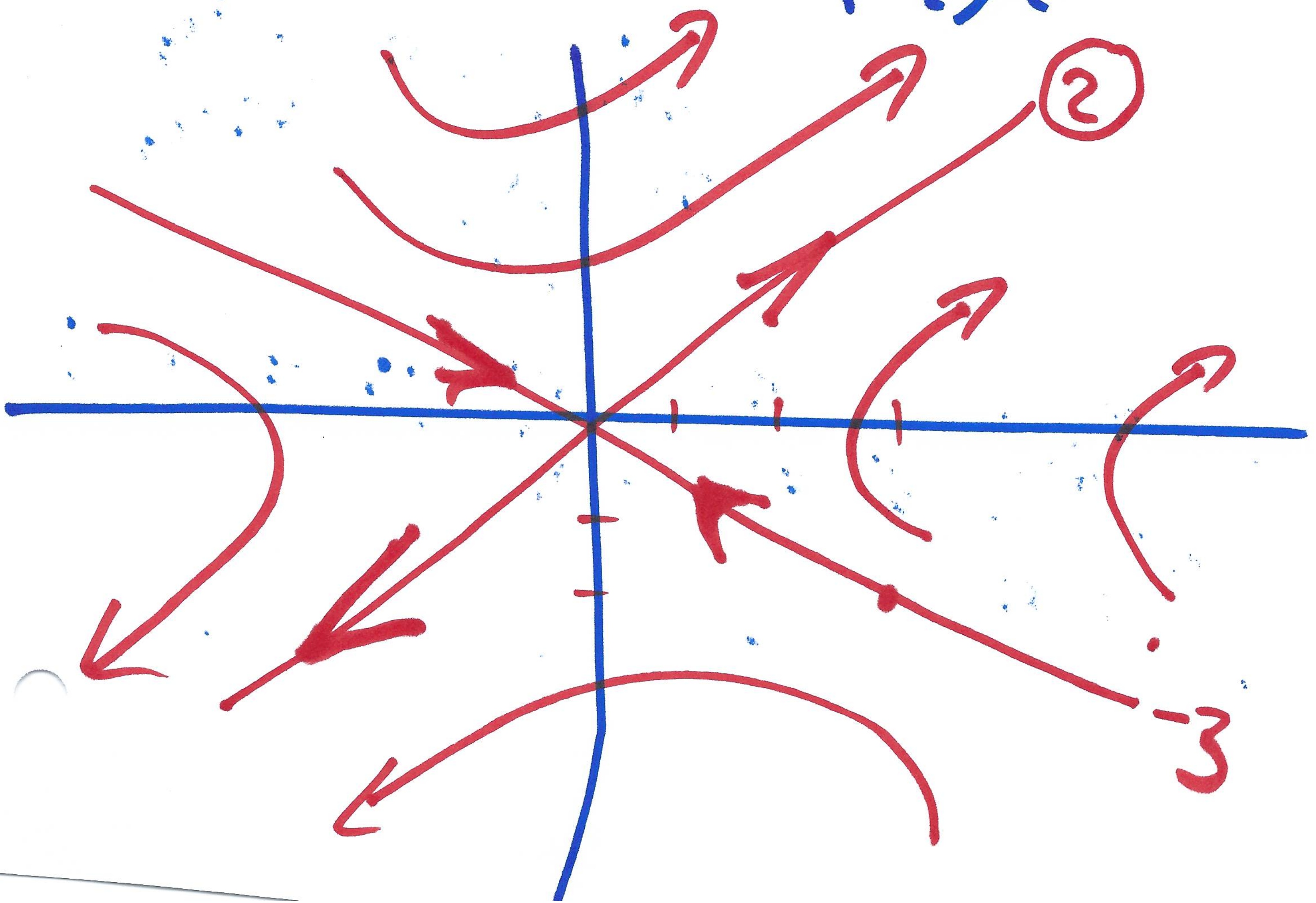
$\lambda_2 = -3; \ v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$c_1 + 3c_2 = x_0$

$c_1 - 2c_2 = y_0$

$c_2 = \dfrac{x_0 - y_0}{5}$

$$x(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 3 \\ -2 \end{pmatrix}e^{-3t}$$

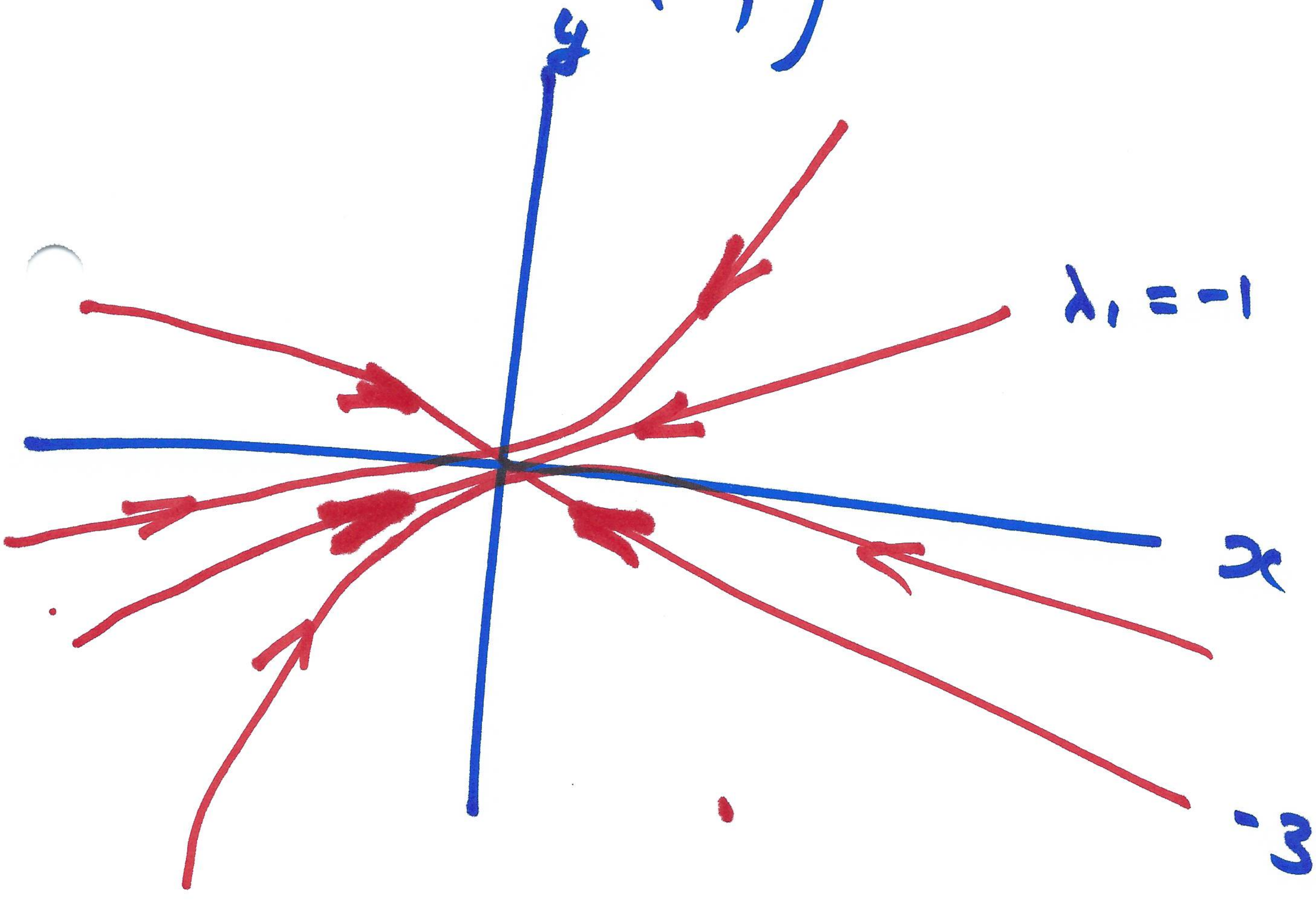


P

$$\dot{x} = \begin{pmatrix} -2 & 2 \\ 1/2 & -2 \end{pmatrix} x$$

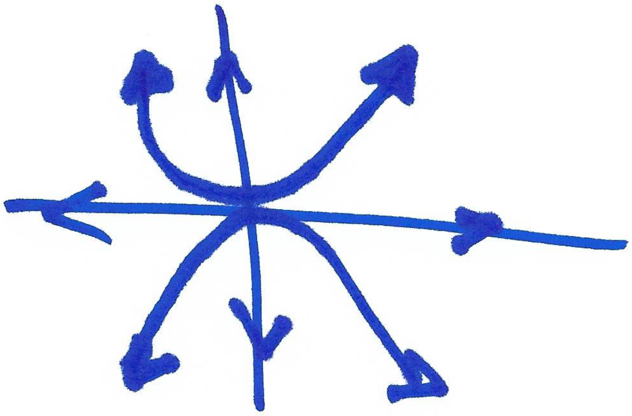
$\lambda_1 = -1; \ v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -3 \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



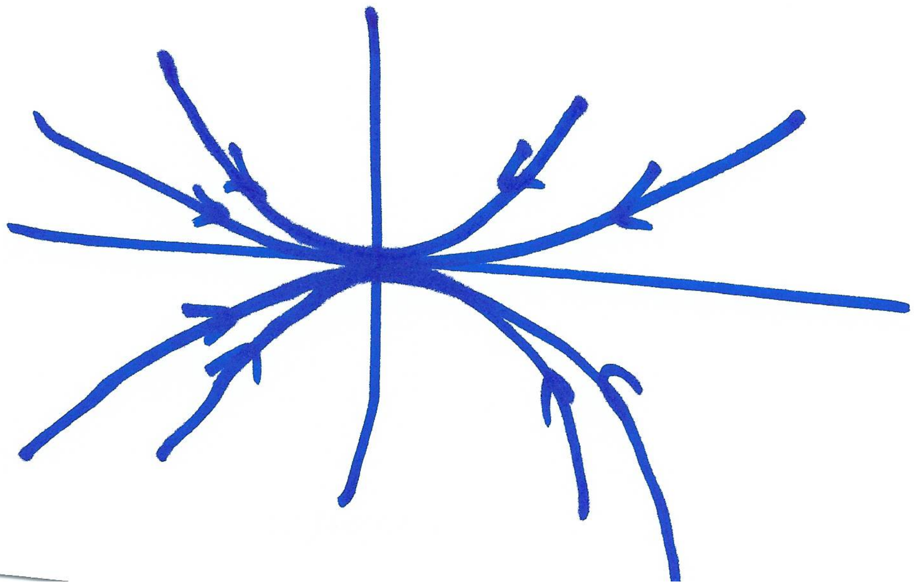
R

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \underline{x}$$



$$\underline{\dot{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \underline{x}$$

$$\underline{x}(t) = \dot{x}_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$



$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}-1 & 3\\ 2 & 0\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

$$\text{Det}(A - \lambda I) = (-1-\lambda)(-\lambda) - 6 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm 5}{2} = -3;\ 2$$

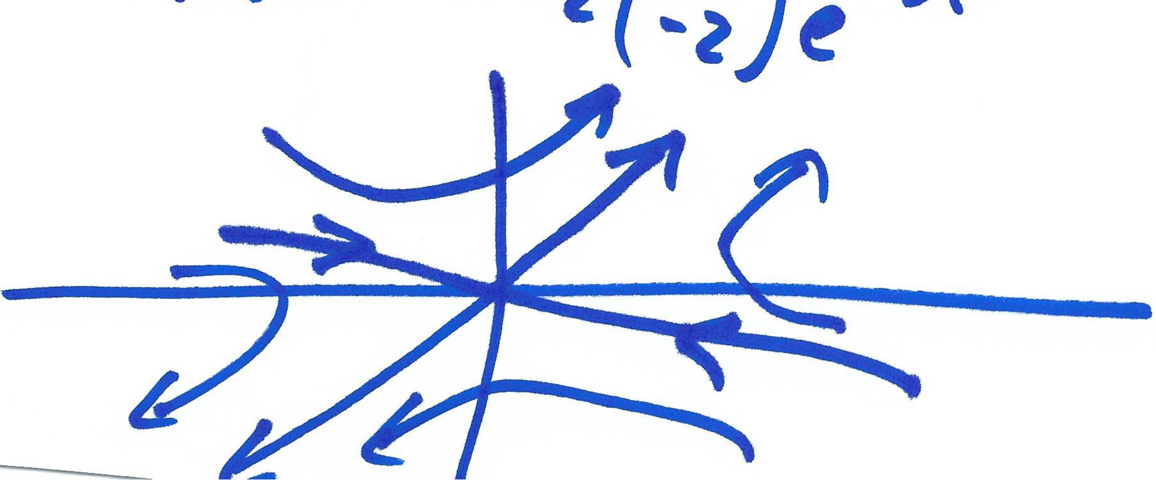
$$-a + 3b = 2a; \quad v_1 = \begin{pmatrix}1\\ 1\end{pmatrix} \quad \lambda_1 = 2$$

$$-c + 3d = -3c; \quad 3d = -2c$$

[illegible] $\begin{pmatrix}3\\ -2\end{pmatrix} = v_2$

$\lambda_2 = -3$

$$y(x) = c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}3\\ -2\end{pmatrix}e^{-3t}$$



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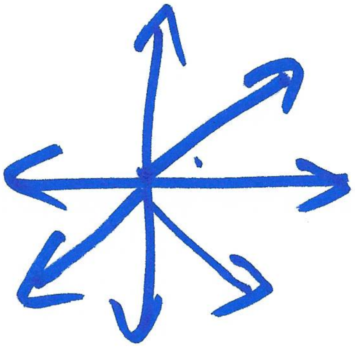
A

Example



$$\frac{d}{dt}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

$$A=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$\lambda_1 = 2 \quad v_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\lambda_2 = 2 \quad v_2 = \begin{pmatrix}0\\1\end{pmatrix}$

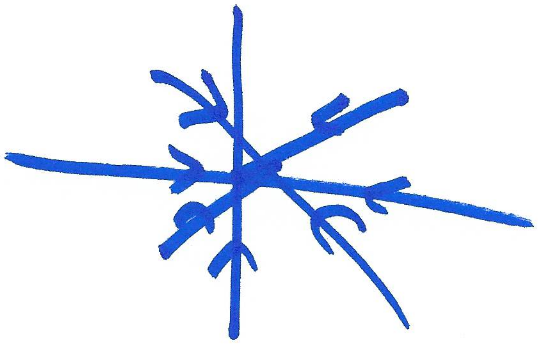
$x(t) = \begin{pmatrix}x_0\\y_0\end{pmatrix}e^{2t}, \ (x_0, y_0)$ is IC

$x(t) = x_0 e^{2t} \Rightarrow e^{2t} = \frac{x(t)}{x_0}$

$y(t) = y_0 e^{2t} = \underline{x(t)}$

$$\dot{\underline{x}} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \underline{x}(t)$$

13

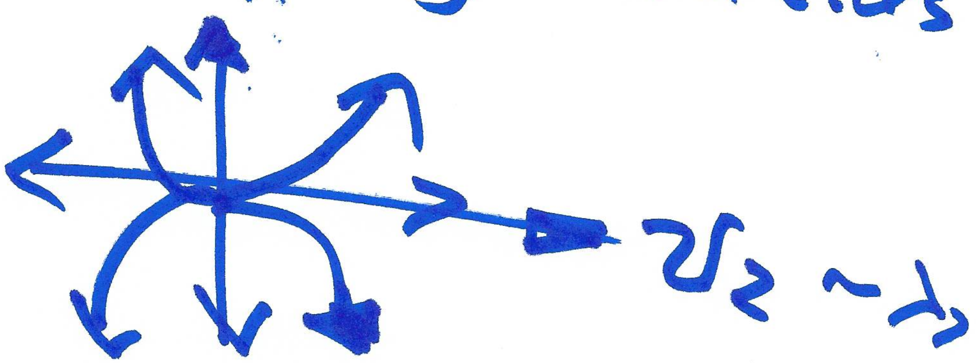


C

Case 1

$\lambda_1 > \lambda_2 > 0$

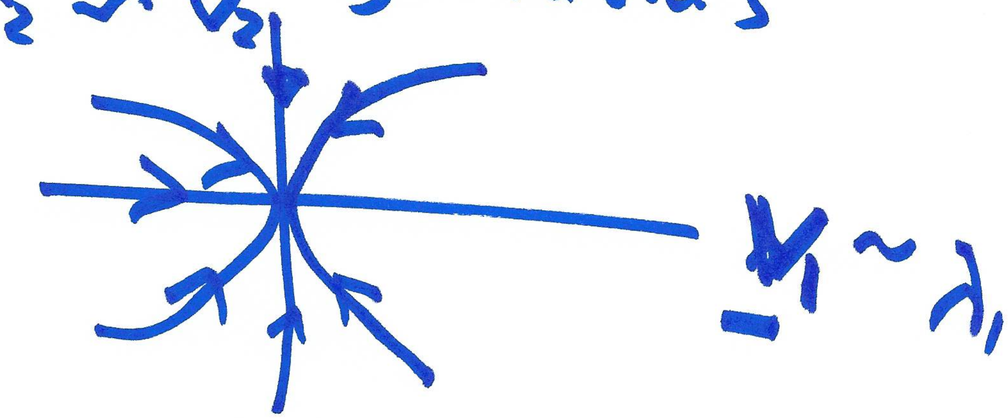
two distinct positive eigenvalues



Case 2

$\lambda_1 < \lambda_2 < 0$

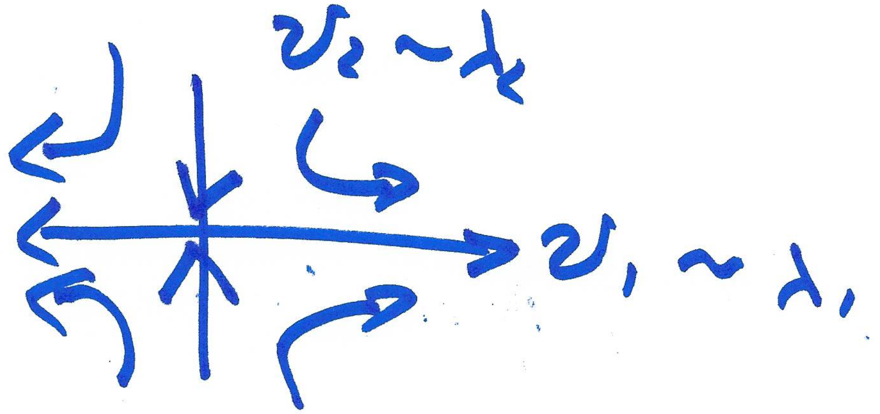
two distinct negative eigenvalues



D

Case 3

- $\lambda_1 > 0, \lambda_2 < 0$

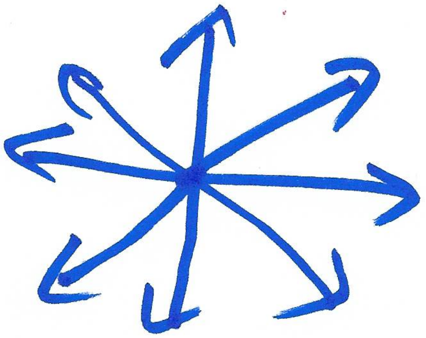


Case 4

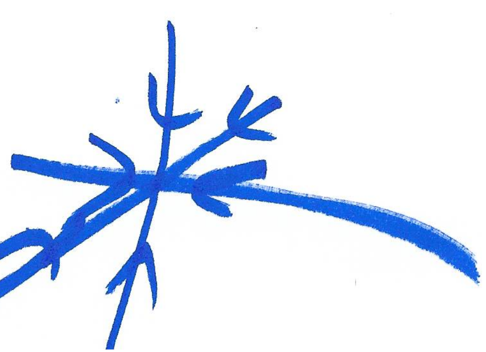
$\lambda_1 = \lambda_2$

$\underline{v}_1 \neq \underline{v}_2$

$\lambda_1 = \lambda_2 > 0$



$\lambda_1 = \lambda_2 < 0$



E

$$\dot{\underline{x}} = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix} \underline{x}$$

$$Det(A - \lambda I) = (-2-\lambda)(2-\lambda)+8$$

$$= (\lambda+2)(\lambda-2)+8 =$$

$= \lambda^2$ [illegible] $+ 4 = 0; \quad \lambda_{1,2} = \pm 2i$

$$\begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2i\begin{pmatrix} a \\ b \end{pmatrix}$$

$$-2a + 4b = 2ia$$

$$4b = (2i+2)a$$

$$\boxed{2\,\boxed{b = (1+i)}\,a}$$

$a = 2$

$b = 1+i$

$$v_1 = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

F

$$\underline{x}(t) = d_1(p\cos\omega t - g\sin\omega t)e^{\gamma}$$

$$+ d_2(p\sin\omega t + g\cos\omega t)e^{\gamma t}$$

$\omega = \mathrm{Im}\,\lambda = 2$

$\gamma = \mathrm{Re}\,\lambda = 0$

$p = \mathrm{Re}\,\underline{v} = \begin{pmatrix}2\\1\end{pmatrix}$

$g = \mathrm{Im}\,\underline{v} = \begin{pmatrix}0\\1\end{pmatrix}$

$$x(t) = d_1\left(\begin{pmatrix}2\\1\end{pmatrix}\cos 2t - \begin{pmatrix}0\\1\end{pmatrix}\sin\omega t\right)$$

$$+ d_2\left(\begin{pmatrix}2\\1\end{pmatrix}\sin 2t + \begin{pmatrix}0\\1\end{pmatrix}\cos 2t\right)$$

$d_1 = 1;\ d_2 = 0$

$$x(t) = \begin{pmatrix}2\cos 2t\\ \cos 2t - \sin\omega t\end{pmatrix}$$

G

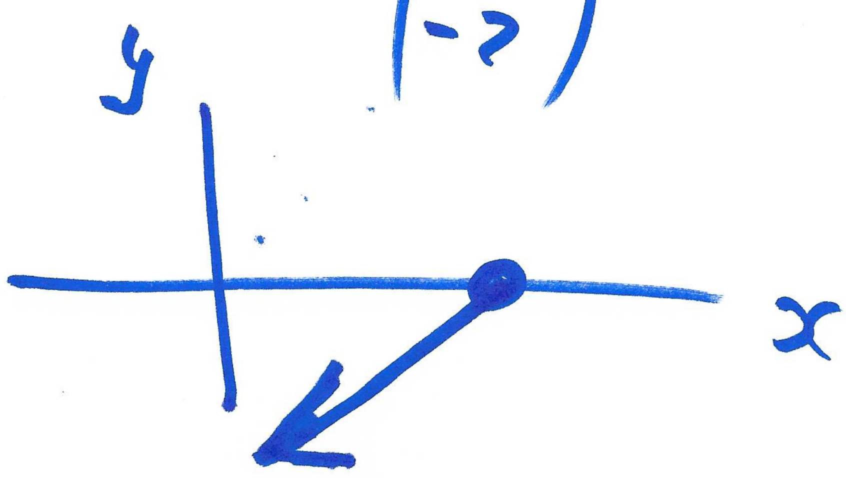
Clock or anti-clock?

$$\dot{x} = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix} \underline{x}$$

$$x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\dot{x}(t=0) = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

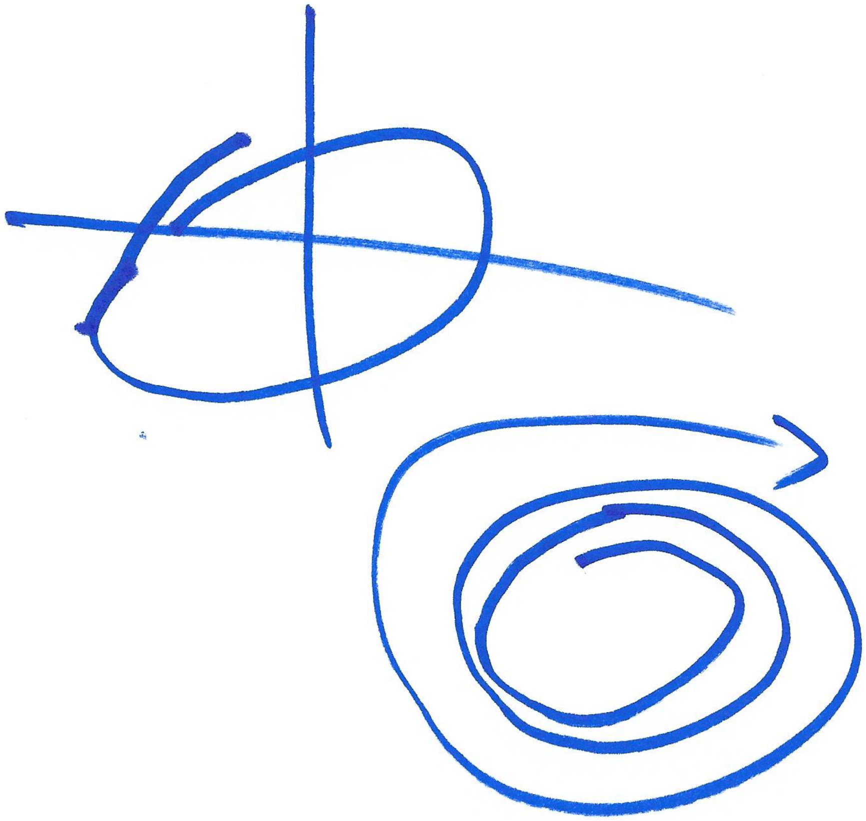


If $\lambda_1 = \lambda_2^*$

and $\mathrm{Re}\,\lambda = 0$

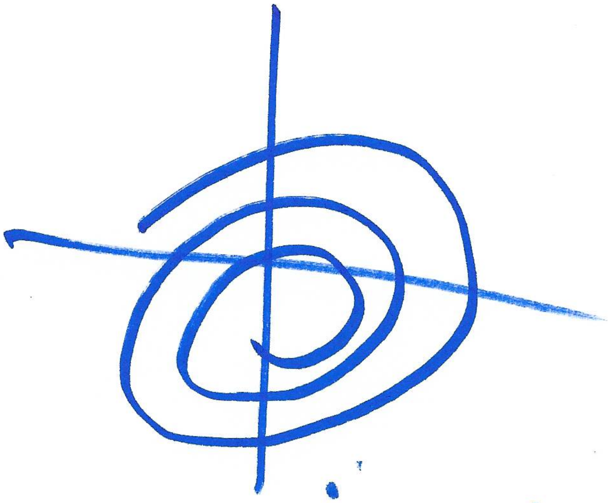
trajectories are ellipses.

$\mathrm{Im}\,\lambda \gg \mathrm{Re}\,\lambda > 0$



I

IF $\mathrm{Re}\,\lambda < 0$



7

$$\dot{\underline{x}} = \begin{pmatrix} 2 & 1 \\ -10 & 0 \end{pmatrix} \begin{pmatrix} \underline{x} \end{pmatrix}$$

$$(2-\lambda)(-\lambda)+10=0$$

$$(\lambda-2)\lambda+10=0$$

$$\lambda^2-2\lambda+10=0$$

$$\lambda_{1,2} = 1 \pm \frac{\sqrt{4-40}}{2}$$

$$= 1 \pm 3i$$

$$2a+b = a+3i\,a$$

~~$(a-3i) = -b$~~

$$a(1-3i) = -b \qquad a=-1 \quad b=1-3i$$

$$v = \begin{pmatrix} -1 \\ 1-3i \end{pmatrix}$$

K

$\omega = \text{Im}\,\lambda = 3$

$\gamma = \text{Re}\,\lambda = 1$

$p = \text{Re}(\underline{v}) = \begin{pmatrix}-1\\1\end{pmatrix}$

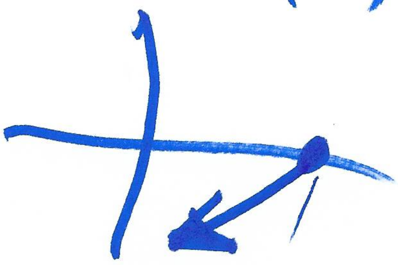
$q = \text{Im}\,v = \begin{pmatrix}0\\-3\end{pmatrix}$

$$x(t) = d_1\left(\begin{pmatrix}-1\\1\end{pmatrix}\cos 3t - \begin{pmatrix}0\\-3\end{pmatrix}\sin 3t\right)e^t$$

$$+\ d_2\left(\begin{pmatrix}-1\\1\end{pmatrix}\sin 3t + \begin{pmatrix}0\\-3\end{pmatrix}\sin 3t\right)e^t$$

Direction? $x_0 = \begin{pmatrix}1\\0\end{pmatrix}$

$$\begin{pmatrix}2 & 1\\-10 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-10\end{pmatrix}$$



Stability of a steady state.

L

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t)$$

IF $\underline{x}(t=0) = 0$ then

$$\underline{x}(t) = 0 \quad \forall t > 0$$

The origin is a "steady state"

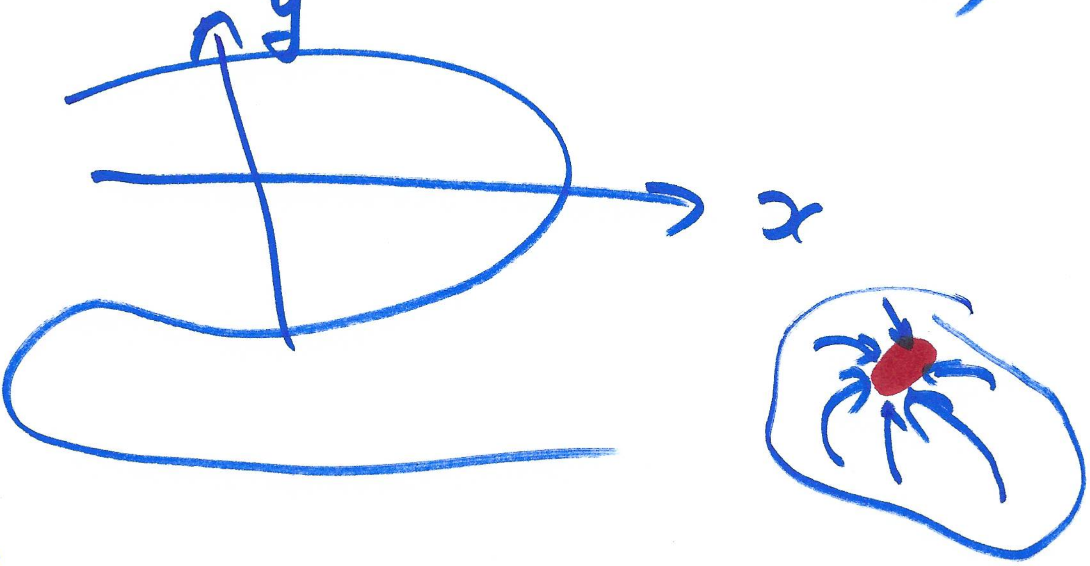
if $\underline{\underline{A}}$ is invertible => $\exists A^{-1}$

- $\underline{x}_s$ is called assymptotically steady state. if for any x_0 sufficiently close to x_s if $x(t=0) = x_0$ =>

$\lim x(t)$

M

$$\left.\begin{aligned} \frac{d}{dt} x(t) &= f(x(t), y(t)) \\ \frac{d}{dt} y(t) &= g(x(t), y(t)) \end{aligned}\right\}$$



- X_u is unstable steady state so that no matter how close to it you start, it is possible to find IC so that $\lim_{t \to} X(t)$

N

IF $\lambda_1 \le \lambda_2 < 0$

and $\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$

then $\begin{pmatrix}0\\0\end{pmatrix}$ is an assymptotically stable state for

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

IF λ_1 is complex and

$\lambda_1 = \lambda_2^*$ and

$\mathrm{Re}(\lambda_1) < 0$ then $\begin{pmatrix}0\\0\end{pmatrix}$

is an assymptotically

$x(t) = -3 + u(t)$ P

$y(t) = 1 + v(t)$

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}2\\ 4\end{pmatrix}$$

$= 0$

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}$$

$(1-\lambda)(-1-\lambda)-1 = 0$

$(\lambda-1)(\lambda+1)-1 = 0 \quad \lambda^2 = 2$

$\lambda = \pm\sqrt{2}$

Q

$$\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \sqrt{2}\begin{pmatrix}a\\b\end{pmatrix}$$

$$a + b = \sqrt{2}\,a$$

$$\boxed{b = (\sqrt{2}-1)}\,a$$

$$\underline{v}_1 = \begin{pmatrix}1\\\sqrt{2}-1\end{pmatrix} \rightsquigarrow \lambda_1 = \sqrt{2}$$

$$\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} = -\sqrt{2}\begin{pmatrix}c\\d\end{pmatrix}$$

$$c + d = -\sqrt{2}\,c$$

$$\boxed{c}\,(1+\sqrt{2}) = \boxed{-}\,d \qquad v_2 = \begin{pmatrix}1+\sqrt{2}\\-1\end{pmatrix}$$